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Avoiding the axiom of choice in general category theory☆

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Abstract

The notion of anafunctor is introduced. An anafunctor is, roughly, a "functor defined up to isomorphism". Anafunctors have a general theory paralleling that of ordinary functors; they have natural transformations, they form categories, they can be composed, etc. Anafunctors can be saturated, to ensure that any object isomorphic to a possible value of the anafunctor is also a possible value at the same argument object. The existence of anafunctors in situations when ordinarily one would use choice is ensured without choice; e.g., for a category which has binary products, but not specified binary products, the anaversion of the product functor is canonically definable, unlike the ordinary product functor that needs the axiom of choice. When the composition functors in a bicategory are changed into anafunctors, one obtains ana-bicategories. In the standard definitions of bicategories such as the monoidal category of modules over a ring, or the bicategory of spans in a category with pullbacks, and many others, one uses choice; the anaversions of these bicategories have canonical definitions. The overall effect is an elimination of the axiom of choice, and of non-canonical choices, in large parts of general category theory. To ensure the Cartesian closed character of the bicategory of small categories, with anafunctors as 1-cells, one uses a weak version of the axiom of choice, which is related to A. Blass' axiom of Small Violations of Choice (1979).

0. Introduction

In Category Theory, there is an underlying principle according to which the right notion of "equality" for objects in a category is isomorphism. Let me refer to the principle as the *principle of isomorphism*. According to the principle of isomorphism, any object isomorphic to a given one should be able to serve the same categorical purposes as the given one. Of course, the principle of isomorphism may be read as a limitation on what properties of objects are to be considered in category theory; but

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the principle also carries with it the assertion that by so restricting the properties of objects, we are not losing any essential element of the situation.

Therefore, when singling out an object with a certain property, we should be content with determining the object up to isomorphism only. Indeed, the categorical operations defined by universal properties (products, exponentials, etc.) determine the object-parts of their values at given arguments only up to isomorphism. The idea behind the notion of *anafunctor*, the main new concept in this paper (see 2.1 (i)–(v) below; a reference of the form m.n (…) is to item n (…) in Section m) is that the same principle should extend to values of functors: their object-values are to be determined up to isomorphism only.

General category theory in its usual form does not quite live up to the principle of isomorphism; the ubiquitous use of the Axiom of Choice (AC) in general category theory is a related fact. A simple example is at hand when, for a category $\boldsymbol{C}$ having binary products of objects, we pass to the consideration of "the" product functor $P = (\ ) \times (\ ) : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$. The definition of P requires the simultaneous *choice* of a specific product $(A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_1} B)$ corresponding to each pair (A, B) of objects. To be sure, in most *examples* of a category $\boldsymbol{C}$ such a simultaneous choice can be made without the AC; however, we want to use the product functor in the theory for any category $\boldsymbol{C}$ with binary products, without knowing anything further about $\boldsymbol{C}$. Whether or not an *explicit* choice of products is available, something of the *canonicity* of the resulting entity (functor) is lost when we make a *particular* choice of products. Actually, talking about *the* product functor becomes imprecise; there are, in general, many possible product functors.

The general form of the above type of use of the AC is in taking "the" adjoint of a functor on the basis of the representability of a family of Set-valued functors derived from the given functor. Every time we use the Adjoint Functor Theorem to get an adjoint, we use the AC in the described manner.

There are similar violations of canonicity and attendant uses of the AC in the definitions of various concrete monoidal categories, and higher dimensional categorical objects.

In this paper, I propose a revision of the notion of functor, that of anafunctor, and consequent revisions of certain higher dimensional concepts, that makes possible a theory based more thoroughly on canonical constructions than ordinary category theory, and specifically, that rectifies the violations described above of the principle of isomorphism. The revisions are non-intrusive in the sense that category theory with anafunctors is of the same general shape as with ordinary functors. It seems that there is no limitation of the applicability of anafunctors in any context where functors are used. The resulting theory avoids Choice to a large extent (although not completely; see below), and still has the same general form as classical general category theory. If one employs the full Axiom of Choice, the new theory reduces to the classical one. Without the Axiom of Choice, we have a product *ana*functor $P = (\ ) \times (\ ) : \boldsymbol{C} \times \boldsymbol{C} \to \boldsymbol{C}$ *defined canonically* on the basis of $\boldsymbol{C}$ having binary products. The adjoint of a(n) (*ana*)functor, an *ana*functor, is given canonically once the condition mentioned above on representability is fulfilled. Anafunctors have natural transformations, which are the arrows of a category as usual; categories with anafunctors and natural transformations form a bicategory. We have *ana*bicategories, *ana*monoidal categories, with basic theories similar in outline to those of their usual counterparts.

Whereas "anafunctor" is a generalization of "functor" a certain specialization of the notion of "anafunctor", "*saturated anafunctor*" is the one that should be regarded as the finished form of the concept; an ordinary functor is (usually) not a saturated anafunctor. Saturated anafunctors (1.1 (vi)) satisfy the analog of Leibniz's principle of substitutability of equal for equal: if an object in the codomain category of the saturated anafunctor is isomorphic to a value of the anafunctor, then it is itself a value of the anafunctor at the same argument, "in a uniquely determined way ". It turns out that saturated anafunctors are sufficient; there is a canonical way of "saturating" any anafunctor, the result of which, a saturated anafunctor, is isomorphic (via a canonical natural isomorphism) to the given anafunctor.

The most important difference between using anafunctors and using functors is a result of the fact that the category Ana $(\boldsymbol{X},\boldsymbol{A})$ of (small) anafunctors between two fixed small categories $\boldsymbol{X}$ and $\boldsymbol{A}$ is not small (unless $\boldsymbol{X}$ or $\boldsymbol{A}$ is empty). However, under the assumption of a certain weak consequence, here called the Small Cardinality Selection Axiom (SCSA), of the Axiom of Choice, Ana $(\boldsymbol{X},\boldsymbol{A})$ is equivalent (in fact, in the strong sense) to a small category. Thus, the SCSA ensures the Cartesian closed character of the bicategory of small categories with anafunctors and natural transformations (with "Cartesian closed" meant in the natural bicategorical sense). The SCSA is closely related to Blass' axiom [3] of Small Violations of Choice (SVC), another weak choice principle.

There is a well-known and important approach to category theory relative to a largely arbitrary topos. See [2, 9, 16, 17]. The theory uses the formalism of indexed categories [9, 17], or alternatively and essentially equivalently, that of fibrations [1, 2]. Category theory done internally in $\mathscr{E}$ is a part of indexed category theory over $\mathscr{E}$. Indexed category theory over $\mathscr{E}$ may use the axiom of choice externally. For instance, in [17], a form of the Initial Object Theorem is proved, and from this, an appropriate form of the Adjoint Functor Theorem is inferred, by the same kind of use of the AC as the one that goes into constructing the product functor mentioned above.

The approach of the present paper is, in a sense, orthogonal to that of indexed category theory: neither approach does what the other does, but they can be combined to work together. When a topos lacks the necessary AC, the product functor mentioned above for an internal category with products (where the mere existence of products, rather than their specifiability, is assumed internally) does not exist internally, and will not exist for the externalization, an indexed category, of the internal category. However, the present paper's approach will provide an internal anafunctor in place of the product functor without assuming Choice in the topos. In fact, the development of the present paper, can be relativized to any topos. In [15], anafunctor theory will be put into the context of indexed category theory over a topos, and a connection will be established with stacks and stack completions. It will be shown that a suitable variant of the SCSA, one that is equivalent to saying that internal categories have internal stack-completions, will ensure that the bicategory of internal categories, internal anafunctors and natural transformations is Cartesian closed.

The present paper is only the beginning of the development of "anafunctor theory". Let me briefly indicate an area of category theory where anafunctors are relevant. This is the general (or universal) algebra of structured categories. The usual kinds of structured categories (lex categories, regular categories, (elementary) toposes (in this case, use only isomorphism 2-cells), and many more) form *locally finitely presentable bicategories*. The latter have a theory formally similar to that of locally finitely presentable categories of [6]. This theory has only partly been codified at the present time, but various key elements of it, such as the theory of bicategorical (indexed or weighted) limits (see, e.g., [18]), have been clarified. The sequel [13] will deal with locally finitely presentable bicategories and related matters by employing anafunctors, giving more canonical answers to existence questions than the usual theory; and avoiding the AC. I now give two indications, to be worked out in [13], why anafunctors are useful for a "canonical" version of the general algebra of structured categories.

One may maintain that, when dealing with a category $\boldsymbol{C}$ with finite products, it is not necessary to invoke the product functor $(\ )\times(\ ):\boldsymbol{C}\times\boldsymbol{C}\to\boldsymbol{C}$; after all, all that this does is to pick a particular product for each argument-pair, and we have the experience that in most cases this is not necessary. However, if we want to make the theory of categories with finite products (fp categories) a part of the algebra of structured categories along the lines hinted above, then the product functor is forced on us. In this theory, $\boldsymbol{C}$ induces a functor (a restricted representable functor) $\dot{\boldsymbol{C}}:\mathrm{FP}_{\mathrm{f}}^{\mathrm{op}}\to\mathrm{Cat}$ on the opposite of the bicategory FP_{f} of the finitely presentable fp categories to the bicategory Cat of small categories, and the above product functor is the value of $\dot{\boldsymbol{C}}$ at the 1-cell $[X]\to[Y,Z]$ in FP_{f}; here, $[X]$ is the fp category freely generated by the object X, similarly for $[Y,Z]$, and the arrow is induced by the mapping $X\to Y\times Z$. (The mapping $\boldsymbol{C}\to\dot{\boldsymbol{C}}$ is the basic identification of the objects of a locally finitely presentable bicategory with a Cat-valued functor. The reader will be familiar with the one-dimensional analog of the described constructions; replace FP_{f} will $\mathrm{Ring}_{\mathrm{f}}$, the category of finitely presentable commutative rings with 1, replace Cat by Set, take a ring R in place of $\boldsymbol{C}$, take $\times$ to be the multiplication in R, and the above with refer to the multiplication-operation $(\ )\cdot(\ ):R\times R\to R$.) In brief, the point of view of the bicategorical algebra of structured categories necessitates the consideration of something like the product functor. We have mentioned that anafunctor theory is capable of providing the needed entity in a canonical fashion.

Another example for the use of anafunctors is as follows. Consider the notion of the free structured category $\mathscr{F}(\boldsymbol{G})$ of a given kind generated by the graph $\boldsymbol{G}$. For the sake of a convincing example, let us talk about categories with finite limits and finite colimits (without any further restriction) as the given kind. Suppose $\boldsymbol{G}$ is a finite graph. In this case, $\mathscr{F}(\boldsymbol{G})$ has an explicit description, consisting of iterated formal limits and colimits, starting with the generators; in particular, certainly, there is no need for

Choice in the construction of $\mathscr{F}(\boldsymbol{G})$. (Andre Joyal has recently given a beautiful theory of just this free construction, and its enriched generalizations.) However, to verify the universal property of $\mathscr{F}(\boldsymbol{G})$, against *all* maps $\varphi : \boldsymbol{G} \to \boldsymbol{C}$ into a category $\boldsymbol{C}$ with finite limits and colimits, in the usual theory we do need some form of the AC. In fact, we are required to construct a functor $F : \mathscr{F}(\boldsymbol{G}) \to \boldsymbol{C}$ preserving finite limits and colimits and satisfying the initial conditions given by φ. The construction of F requires a series of choices of limits and colimits in $\boldsymbol{C}$, which cannot be done without Choice. The use of an anafunctor in place of F eliminates the need of the AC, and in fact makes F canonical.

Of course, the last example is a crucial one for the general algebra of structured categories; in this theory, we would not want to do without free objects such as $\mathscr{F}(\boldsymbol{G})$.

Let me turn to remarks on the set-theory used in the paper.

The set-theoretic foundations used in this paper are "minimal", and probably the reader will have no problem following the paper even if he skips these (brief) preliminaries.

We work in a constructive set-theory with sets and classes. For the sake of definiteness, we take as our foundations the Gödel–Bernays (G–B) axioms for sets and classes [7], without the AC, and without the Axiom of Regularity (Foundation), and we employ intuitionistic predicate logic to deduce consequences of the axioms. (We could accommodate ur-elements, but to do so would require some explanations that we do not want to give; thus, all things in our theory are classes, and some classes (precisely those that are elements of some class) are sets; the axiom of extensionality is assumed in an unrestricted form.) We do not use Grothendieck universes.

The use of the adjective "small" will, as usual, signify that the entity it qualifies is a set. Thus, a small class is the same thing as a set.

A *category* $\boldsymbol{A}$ is given by a *class* of objects $\mathrm{Ob}(\boldsymbol{A})$, and a *class* $\mathrm{Arr}(\boldsymbol{A})$ of arrows, with further data as usual. Thus, we do not make the blanket assumption that a category has small hom-sets; if it does, it is said to be *locally small*. A *small* category has both $\mathrm{Ob}(\boldsymbol{A})$ and $\mathrm{Arr}(\boldsymbol{A})$ sets; of course, $\mathrm{Arr}(\boldsymbol{A})$ being a set implies that $\mathrm{Ob}(\boldsymbol{A})$ is one as well. A small category can be regarded as a single set (e.g., as a tuple $(|\boldsymbol{A}|, Arr(\boldsymbol{A}), \ldots))$, and we may talk about the *class* (and eventually, the *category*) of all small categories.

Note that a category isomorphic to a small category *is* small (by the Axiom of Replacement).

Within G–B, one cannot talk about the category of all functors $\boldsymbol{X} \to \boldsymbol{A}$ for two fixed, but arbitrary categories $\boldsymbol{X}, \boldsymbol{A}$; there are no collections whose members are proper classes. Of course, there is no problem when the categories $\boldsymbol{X}$, $\boldsymbol{A}$ are small, or even when just $\boldsymbol{X}$ is small (since in the latter case functors $\boldsymbol{X} \to \boldsymbol{A}$ are (may be regarded as) sets). However, within the framework of the formal base-theory G–B, we may contemplate *metacategories*; an example is FUN $(\boldsymbol{X},\boldsymbol{A})$, the metacategory of all functors $\boldsymbol{X} \to \boldsymbol{A}$ and natural transformations. Formally, a metacategory is given by predicates (formulas) $\mathrm{Ob}(X,\vec{P})$, $\mathrm{Arr}(f,\vec{P})$, $\mathrm{Dom}(f,X,\vec{P})$, $\mathrm{Codom}(f,X,\vec{P})$, $\mathrm{Comp}(f,g,h,\vec{P})$ of the base-theory (in our case, G–B), with the free variables shown, all ranging over *classes*, together with the assumption that, for a fixed value of the parameters $\vec{P}$, the obvious equivalents of the category axioms (which become first order formulas, having only $\vec{P}$ as free variables, built up of the given predicates) hold. The said assumption may be a consequence of an assumption $C(\vec{P})$ on the parameters $\vec{P}$. In the case of Fun $(\boldsymbol{X},\boldsymbol{A})$, $\vec{P}$ is $\boldsymbol{X},\boldsymbol{A}$ [although a category $\boldsymbol{X}$ is given by classes $|\boldsymbol{X}|$, Arr $(\boldsymbol{X})$, ... , these can be combined, although somewhat artificially, into a single class; if we do not want to do this, $\vec{P}$ will be a longer tuple, listing all the data-classes of both categories $\boldsymbol{X},\boldsymbol{A}$], and $C(\vec{P})$ is the assumption that $\boldsymbol{X}$, $\boldsymbol{A}$ are indeed categories. Of course, the idea of a metacategory is just one instance of a family of meta-concepts similarly fashioned from a formal concept such as "category". One can, e.g., talk about CAT, the meta-2-category of all categories, functors and natural transformations; Cat is the 2-category of *small* categories, functors and natural transformations.

Let me note that I will usually drop the "meta" prefix from constructs such as meta-functor, meta-natural transformation, etc.

Although [5] does not mention a formalized base-theory in which the exposition is made, it is rather clear that a class-set theory is meant such as G–B; no universes are employed. On the other hand, the explicit base-theory in [11] is Zermelo–Fraenkel (ZF) set theory, a theory of sets without class-variables. One universe (a set U with appropriate properties) is used, and the word 'small" is reserved for members of U. [11] uses "class" in a somewhat non-standard manner; classes in [11] are non-small *sets*.

Our base-theory is like that of [5]; in particular our categories and the categories in [5] may be large (classes); the word "small" is used here in agreement with [5]; however, [5] does not mention "metacategories". The "metacategories" of [11] are our categories. Our metacategories are introduced on the same principle as those of [11], but the difference in the base-theories makes the meanings of the term different.

The use of the prefix "ana" has been suggested by Dusko Pavlovic. He noted the use of "pro-" in category theory (profunctor, proobject), and noted that in biology, the terms "anaphase" and "prophase" are used in the same context.

At a time when the work on this paper had essentially been completed, Robert Paré told me that he had had related ideas in the 1970's, and he had lectured about them at a meeting in New York in 1975, although he had not published his work.

Some time after the first version of this paper was written, I was informed that a special case of the notion of anafunctor, and of the notion of natural transformation of anafunctors, the case when the domain category is **1**, the terminal category, have been introduced in [10], under the name of "clique" and morphism of cliques. In [10], cliques are used for certain special purposes; beyond the definition of cliques and their morphisms, there is essentially no overlap between [10] and this paper. For more precise references, see toward the end of Section 1 of this paper.

1. Anafunctors

Let $\boldsymbol{X}$ and $\boldsymbol{A}$ be categories. An *anafunctor F with domain $\boldsymbol{X}$ and codomain $\boldsymbol{A}$*, in notation $F:\boldsymbol{X}\xrightarrow{\text{a}}\boldsymbol{A}$, or just simply $F:\boldsymbol{X}\to\boldsymbol{A}$, is given by the following data 1. (i), (ii) and conditions 1. (iii)–(v):

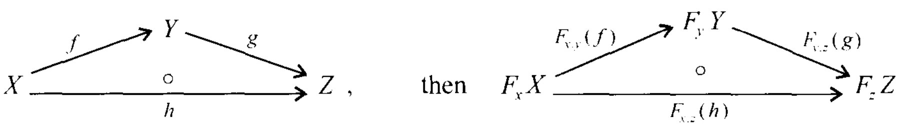
1. (i) A class $|F|$, with maps $\sigma : |F| \to \mathrm{Ob}(\boldsymbol{X})$ ("source"), $\tau : |F| \to \mathrm{Ob}(\boldsymbol{A})$ ("target"). $|F|$ is the class of *specifications*; $s \in |F|$ "specifies the value $\tau(s)$ at the argument $\sigma(s)$". For $X \in \boldsymbol{X}$ (that is, $X \in \mathrm{Ob}(\boldsymbol{X})$), we write $|F|\ X$ for the class $\{s \in |F| : \sigma(s) = X\}$, and $F_s(X)$ for $\tau(s)$; the notation $F_s(X)$ presumes that $s \in |F|\,X$.

(ii) For each X, $Y \in \boldsymbol{X}$, $x \in |F|\,X$, $y \in |F|\,Y$ and $f : X \to Y (\in \mathrm{Arr}(\boldsymbol{X}))$, an arrow $F_{x,y}(f) : F_x(x) \to F_y(Y)$ in $\boldsymbol{A}$.

(iii) For every $X \in \boldsymbol{X}$, $|F|\,X$ is inhabited.

(iv) For all $X \in \boldsymbol{X}$ and $x \in |F|\,X$, we have $F_{x,x}(1_X) = 1_{F_x X}$.

(v) Whenever X, Y, $Z \in \boldsymbol{X}$, $x \in |F|\,X$, $y \in |F|\,Y$, $z \in |F|\,Z$, and



(a circle in a diagram means that the diagram commutes), i.e.,

$$F_{x,z}(gf) = F_{y,z}(g) \cdot F_{x,y}(f).$$

With any given $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$, we put $|F|\,(X, A) \underset{\mathrm{def}}{=} \{x \in |F|\,X : F_x(X) = A\}$.

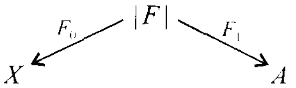
The anafunctor $F : \boldsymbol{X} \to \boldsymbol{A}$ is *locally small* if all the classes $|F|(X, A)$ $(X \in \boldsymbol{X},\ A \in \boldsymbol{A})$ are sets. It is *weakly small* if the classes $|F|\,X$ are all small $(X \in \boldsymbol{X})$; thus, "weakly small" implies "locally small". Finally, F is *small* iff it is weakly small, and the cateogry $\boldsymbol{X}$ is small. Notice that if F is small, then it is given by a *set* of data, beyond the data for $\boldsymbol{A}$; in particular, we may consider the *class of* all small anafunctors with a fixed codomain $\boldsymbol{A}$, an arbitrary (not necessarily small) category.

If $F : \boldsymbol{X} \xrightarrow{\mathrm{a}} \boldsymbol{A}$, and $s \in |F|\,X$, $t \in |F|\,X$, then $F_{s,t}(1_X) : F_s X \to F_t X$ is an isomorphism, with inverse $F_{t,s}(1_X)$. In particular, the *value* of F at X, $F_s(X)$, is determined up to isomorphism.

Any (ordinary) functor $F : \boldsymbol{X} \to \boldsymbol{A}$ is, essentially, an anafunctor, by putting $|F| = \mathrm{Ob}(\boldsymbol{X})$, $\sigma(X) = X$, $\tau(X) = F(X)$ (thus $|F|\,X = \{X\}$), with the obvious specification of the rest of the structure.

A more abstract way of defining the concept is as follows. A *discrete* category is one in which all arrows are identities; an *antidiscrete* category is one in which for any pair (U, V) of objects, there is exactly one arrow $U \to V$. A *discrete* (*antidiscrete*) opfibration is one in which every fiber is a discrete (antidiscrete) category. A discrete opfibration is a functor $G : \boldsymbol{S} \to \boldsymbol{B}$ such that for any $a : A \to B$ in $\boldsymbol{B}$ and $S \in G^{-1}(A)$, there is exactly one arrow $s : S \to T$ with some $T \in G^{-1}(B)$ such that $G(s) = a$; an antidiscrete opfibration is a functor $G : \boldsymbol{S} \to \boldsymbol{B}$ such that for any $a : A \to B$ in $\boldsymbol{B}$, $S \in G^{-1}(A)$ and $T \in G^{-1}(B)$, there is exactly one arrow $s : S \to T$ such that $G(s) = a$. Now,

1*. *An anafunctor $F: \boldsymbol{X} \to \boldsymbol{A}$ may be given by a span*

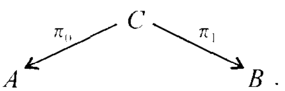


(1)

of functors in which F_0 is an antidiscrete opfibration that is surjective on objects.

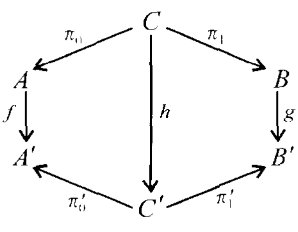
Indeed, with $F: \boldsymbol{X} \to \boldsymbol{A}$ being an anafunctor in the original sense, we let $|F|$ be the category whose object-class is what was $|F|$ above, whose arrows $f: x \to y$ are the same as arrows $f: \sigma(x) \to \sigma(y)$ in $\boldsymbol{X}$, with the obvious composition; F_0 is the obvious forgetful functor (clearly an antidiscrete opfibration); F_1 maps s to $\tau(s)$ and $f: x \to y$ to $F_{x,y}(f)$. Conversely, if we have an anafunctor in the new sense, we put the object-class of $|F|$ for $|F|$ in the old sense, $\sigma(x) = F_0(x)$, $\tau(x) = F_1(x)$, and for $f: X \to Y$ in $\boldsymbol{X}$, x, $y \in |F|$ with $F_0(x) = X$, $F_0(y) = Y$, we put $F_{x,y}(f) = F_1(\hat{f})$ for the unique $\hat{f}: x \to y$ for which $F_0(\hat{f}) = f$.

2. Example. Suppose the category $\boldsymbol{A}$ has binary products; that is, for every $A, B \in \boldsymbol{A}$, there is at least one product diagram



(1)

Then we have the following anafunctor $P: \boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$. $|P|$ consists of all product diagrams of the form (1); for s the diagram in (1), $\sigma(s) = (A, B)$ and $\tau(s) = C$. In the formulation of **1***, $|P|$ is the category of all product diagrams, where arrows are given as in (2) below. In other words, for $(A, B) \in \boldsymbol{A} \times \boldsymbol{A}$, $|P|((A, B))$ is the class of all product diagrams (1), with the given A, B, but all other data variable; for $s \in |P|((A, B))$ given by (1), $P_s((A, B)) = C$. For $s \in |P|((A, B))$ given by the data in (1), and $s' \in |P|((A', B'))$, given by data as in (1) but primed, and for $(f, g): (A, B) \to (A', B')$ ($\in \mathrm{Arr}(\boldsymbol{A} \times \boldsymbol{A})$), $P_{s,s'}((f, g)): C \to C'$ is the unique h making the diagram



(2)

commute; the universal property of the product consisting of the primed data ensures that $P_{s,s'}((f, g))$ is well-defined. It is fairly clear that conditions 1.(iii)–(v) are all satisfied.

The above-defined P is the *product-anafunctor* for the category $\boldsymbol{A}$, "replacing" the product-functor $(A, B) \to A \times B$. Whereas the definition of the latter requires a non-canonical *choice* of a particular product $A \times B$ for each pair (A, B) of objects, and thus, in general, for its definition, the product-functor needs the AC, the product-anafunctor does not involve any non-canonical choice, in particular, it does not need the AC. Of course, it is still to be demonstrated that the product-anafunctor does enough of the job of the product-functor, for it to be a reasonable replacement. At any rate, it will turn out (see below) that if the product-functor exists, then the product-anafunctor is *isomorphic* to it, by an appropriate notion of (natural) isomorphism.

An anafunctor $F:\boldsymbol{X} \xrightarrow{a} \boldsymbol{A}$ is *saturated* if it satisfies the following additional condition:

1. (vi) (*unique transfer*) Whenever $s \in |F|(X, A)$, and $\mu : A \xrightarrow{\cong} B$ is an isomorphism (in $\boldsymbol{A}$), then there is a unique $t \in |F|(X, B)$ such that $\mu = F_{s,t}(1_X)$.

With F an anafunctor, and continuing with the above notation, if $|F|(X, A)$ is inhabited, then A is a *possible value* of F at the argument X. Note that the possible values of F at a given X form a subclass of an isomorphism class of objects in $\boldsymbol{A}$; if F is saturated, they form a complete isomorphism class.

An anafunctor determines its values at least up to isomorphism; a saturated one determines its values *exactly* up to isomorhism. Among anafunctors, the ordinary functors and the saturated anafunctors represent two extremes; our ultimate goal here is to promote the use of the saturated anafunctors as the ones that stand for the point of view that objects (in this case the values of the anafunctor) should be determined *exactly* up to isomorphism, just as they are when they are determined by a universal property.

2. Example (*continued*). The product anafunctor $P:\boldsymbol{A} \times \boldsymbol{A} \to \boldsymbol{A}$ is saturated, as it is immediately seen.

Note that if $F:\boldsymbol{X} \to \boldsymbol{A}$ is saturated, $X \in \boldsymbol{X}$, $s \in |F|(X, A)$, then for any $B \in \boldsymbol{A}$ we have the bijection

$$
\begin{aligned}
|F|(X, B) &\xrightarrow{\cong} \mathrm{Iso}\,(A, B) \\
t &\longrightarrow F_{s,t}(1_X). \qquad (2')
\end{aligned}
$$

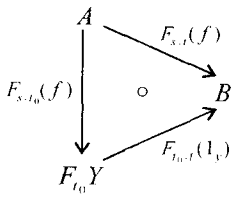
This bijection is not canonical; it depends on the choice of $s \in |F|(X, A)$. Nevertheless, it follows that for a saturated anafunctor $F:\boldsymbol{X} \to \boldsymbol{A}$, if $\boldsymbol{A}$ is locally small, then so is F, and if both $\boldsymbol{X}$ and $\boldsymbol{A}$ are small, then so is F.

Assume $F:\boldsymbol{X} \to \boldsymbol{A}$ is a saturated anafunctor. We have a form of "isomorphic transfer" not only for the values but also for the arguments of F. More precisely,

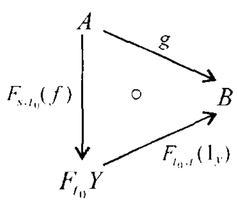
3. *For* $F: \boldsymbol{X} \to \boldsymbol{A}$ *a saturated anafunctor, a pair of isomorphisms* $(f X \xrightarrow{\cong} Y,$ $g: A \xrightarrow{\cong} B)$ *induces a bijection* $|F|(X,A) \xrightarrow{\cong} |F|(Y,B)$, *defined by* $s \mapsto t \Leftrightarrow$ $F_{s,t}(f) = g$.

Let us fix $f: X \xrightarrow{\cong} Y$ and $g: A \xrightarrow{\cong} B$. Let $s \in |F|\ (X,A)$; I claim that there is a unique $t \in |F|\ (Y,B)$ such that $F_{s,t}(f) = g$. Once this is shown, for any $t \in |F|\ (Y,B)$ there is a unique $s \in |F|\ (X,A)$ such that $F_{t,s}(f^{-1}) = g^{-1}$, that is, $F_{s,t}(f) = g$, and the definition above indeed gives a bijection $s \mapsto t$.

Let $t_0 \in |F|\ Y$, $s \in |F|\ (X,A)$, $t \in |F|\ (Y,B)$, and consider the commutative triangle



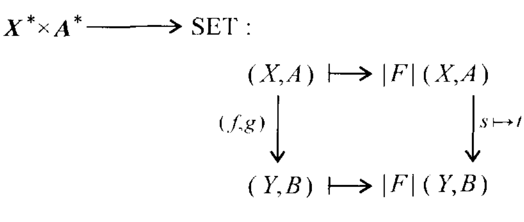
consisting of isomorphisms. It follows that saying that $F_{s,t}(f) = g$ is equivalent to saying that the triangle



commutes. But by 1.(vi), for any $g: A \xrightarrow{\cong} B$, there is a unique t satisfying this latter condition, that is, $F_{t_0,t}(1_Y) = g \circ (F_{s,t_0}(f))^{-1}$.

With $\boldsymbol{X}^*$ denoting the groupoid of all isomorphism in $\boldsymbol{X}$, and similarly for $\boldsymbol{A}^*$, we have,

4. *With* $F: \boldsymbol{X} \to \boldsymbol{A}$ *a saturated anafunctor, the mapping in* 3 *defines a functor* $\boldsymbol{X}^* \times \boldsymbol{A}^* \to$ SET:



Equivalently,

5. *An anafunctor as in* 1* *is saturated iff the induced functor* $|F|^* \to \mathbf{X}^* \times \mathbf{A}^*$ *is a discrete opfibration.*

A *natural transformation* $h: F \to G$ of anafunctors $\mathbf{X} \overset{F}{\underset{G}{\rightrightarrows}} \mathbf{A}$ is given by

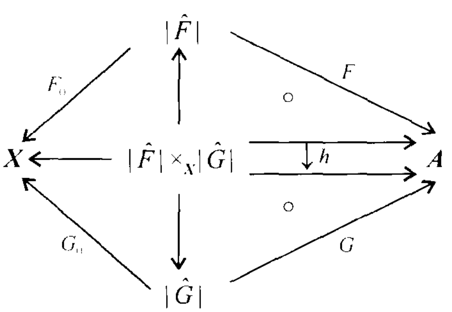
6. (i) a family $\langle h_{s,t}: F_s X \to G_t X \rangle_{X \in \mathbf{X}, s \in |F| X, t \in |G| X}$ of arrows in $\mathbf{A}$ such that

(ii) (*naturality*) for every $f: X \to Y$ in $\mathbf{X}$, and for every $s \in |F| X$, $t \in |G| X$, $u \in |F| Y$, $v \in |G| Y$, the square

$$\begin{array}{ccc} F_s X & \xrightarrow{F_{s,u}(f)} & F_u Y \\ {\scriptstyle h_{s,t}}\downarrow & & \downarrow{\scriptstyle h_{u,v}} \\ G_t X & \xrightarrow[G_{t,v}(f)]{} & G_v Y \end{array} \tag{3}$$

commutes.

An equivalent definition is this. Given anafunctors $(\mathbf{X} \xleftarrow{F_0} |F| \xrightarrow{F} \mathbf{A})$, $(\mathbf{X} \xleftarrow{G_0} |G| \xrightarrow{G} \mathbf{A})$ in the style of 1*, a natural transformation from F to G is the same as a natural transformation h in the usual sense as in the following diagram:



Continuing with the notation of 6, note that if $s, u \in |F| X$, $t, v \in |G| X$, then $h_{u,v}$ is determined by $h_{s,t}$; this is because of the communtativity of

$$\begin{array}{ccc} F_s X & \xrightarrow[\cong]{F_{s,u}(1_X)} & F_u X \\ {\scriptstyle h_{s,t}}\downarrow & & \downarrow{\scriptstyle h_{u,v}} \\ G_t X & \xrightarrow[G_{t,v}(1_X)]{\cong} & G_v X \end{array} \tag{4}$$

Suppose we have a family $\langle (s_i \in |F| X_i, t_i \in |G| X_i) \rangle_{i \in I}$ such that for all $X \in \mathbf{X}$, $X = X_i$ for some $i \in I$. Suppose we have $\langle h_i: F_{s_i}(X_i) \to G_{t_i}(X_i) \rangle_{i \in I}$ such that the naturality condition (3) holds for these data, that is,

$$\begin{array}{ccc} F_{s_i}X_i & \xrightarrow{F_{s_i,s_j}(f)} & F_{s_j}X_j \\ {\scriptstyle h_i}\downarrow & \circ & \downarrow{\scriptstyle h_j} \\ G_{t_i}X_i & \xrightarrow[G_{t_i,t_j}(f)]{} & G_{t_j}X_j \end{array} \qquad (3')$$

for any $i, j \in I$, and $f: X_i \to X_j$.

7. *Under the stated conditions, there is a unique $h: F \to G$ such that $h_{s_i,t_i} = h_i$ for all $i \in I$.*

The rest of the data for h are determined by appropriate instances of the diagram (4).

For any anafunctor $\mathbf{X} \xrightarrow{F} \mathbf{A}$, we have the *identity* natural transformation $1_F: F \to F$, defined by $(1_F)_{s,t} \underset{\text{def}}{=} F_{s,t}(1_X): F_sX \to F_tX$. Naturality of 1_F is a consequence of 1.(v). As a consequence of 7, $h: F \to F$ is equal to 1_F iff $h_{s,s} = 1_{F_sX}$ for all $X \in \mathbf{X}$, $s \in |F|X$.

Composition $k \circ h: F \to H$ of h, k in

$$\mathbf{X} \begin{array}{c} \xrightarrow{F} \\ \downarrow h \; G \\ \longrightarrow \\ \downarrow k \\ \xrightarrow[H]{} \end{array} \mathbf{A}$$

is defined in the expected manner: for $s \in |F|X$, $u \in |H|X$, $(k \circ h)_{s,u}: F_sX \to H_uX$ is the composite of $F_sX \xrightarrow{h_{s,t}} G_tX \xrightarrow{k_{t,u}} H_uX$, with any $t \in |G|X$;

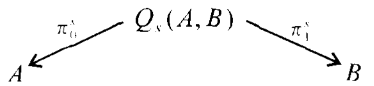
6. (iii) $(k \circ h)_{s,u} \underset{\text{def}}{=} k_{t,u} \circ h_{s,t}$;

for one thing, such t exists; for another, with arbitrary $t, t' \in |G|X$, the commutative diagram

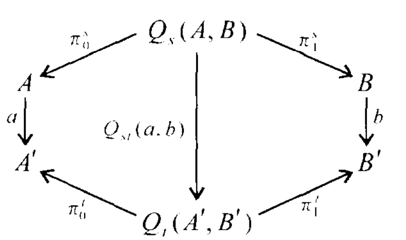
$$\begin{array}{ccc} F_sX & \xrightarrow{1_{F_sX} = F_{s,s}(1_X)} & F_sX \\ {\scriptstyle h_{s,t}}\downarrow & & \downarrow{\scriptstyle h_{s,t'}} \\ G_tX & \xrightarrow[\cong]{G_{t,t'}(1_X)} & G_{t'}X \\ {\scriptstyle k_{t,u}}\downarrow & & \downarrow{\scriptstyle k_{t',u}} \\ H_uX & \xrightarrow[1_{H_uX} = H_{u,u}(1_X)]{} & H_uX \end{array}$$

shows that $(k \cdot h)_{s,u}$ is well-defined (independent of the choice of t). The naturality (6.(ii)) of $k \cdot h$ so defined is seen immediately; and so are the associativity of the composition of natural transformations, and the identity character of the identity natural transformations.

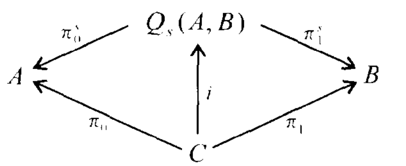
2. Example (*continued*). $Q:\mathbf{A}\times\mathbf{A}\xrightarrow{a}\mathbf{A}$ qualifies as *a* product-anafunctor if, for any $A, B \in \mathbf{A}$, there is a mapping associating with any $s \in |Q|\ ((A,B))$ a product diagram



such that, for any $s \in |Q|\ ((A,B))$, $t \in |Q|\ ((A',B'))$, $a:A \to A'$, $b:B \to B'$, we have that



commutes. Certainly, any product-functor, making a choice of each product, will, as an anafunctor, satisfy the stated condition. But notice that any such Q is isomorphic to P: with $s \in |P|\ ((A,B))$ as in (1), and $t \in |Q|\ ((A,B))$, we can put $h_{s,t}: C \to Q_s(A,B)$ to be the unique isomorphism i that makes



commute; h so defined is an isomorphism $P\xrightarrow{\cong}Q$ as it is easily seen. In particular, if the product-functor *exists*, it is isomorphic to the product-anafunctor (which always exists).

Given categories $\mathbf{A}$, $\mathbf{X}$ with $\mathbf{X}$ small, $\mathrm{Ana}(\mathbf{X},\mathbf{A})$, $\mathrm{Sana}(\mathbf{X},\mathbf{A})$ denote the categories of all *small* anafunctors, respectively *small* saturated anafunctors, $\mathbf{X} \to \mathbf{A}$; arrows are the natural transformations, with composition as given above; $\mathrm{Sana}(\mathbf{X},\mathbf{A})$ is a full subcategory of $\mathrm{Ana}(\mathbf{X},\mathbf{A})$. When $\mathbf{A}$ and $\mathbf{X}$ are both small, we might still have anafunctors $\mathbf{X} \to \mathbf{A}$ that are not small; however, as we said above, all saturated ones are small, and thus belong to $\mathrm{Sana}(\mathbf{X},\mathbf{A})$. We should point out that if $\mathbf{A}$ has an isomorphism class of objects which is not small (a very common occurrence), and $F:\mathbf{X} \to \mathbf{A}$ takes a value in such an isomorphism class, then F cannot be saturated and small at the same time; the category $\mathrm{Sana}(\mathbf{X},\mathbf{A})$ is of importance mainly when both $\mathbf{A}$ and $\mathbf{X}$ are small. Let us also point out that, for small $\boldsymbol{X}$ and $\boldsymbol{A}$, Ana$(\boldsymbol{X},\boldsymbol{A})$, and even Sana$(\boldsymbol{X},\boldsymbol{A})$, cannot be shown to be equivalent to a small category; however, a weak version of the AC will suffice for this last conclusion; see later.

For convenience of expression, we will talk about the metacategories ANA$(\boldsymbol{X},\boldsymbol{A})$, SANA$(\boldsymbol{X},\boldsymbol{A})$ of all anafunctors, resp. saturated anafunctors $\boldsymbol{X} \to \boldsymbol{A}$, with natural transformations as arrows. The notations $\mathrm{ANA}_{ls}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{SANA}_{ls}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A})$, $\mathrm{SANA}_{ws}(\boldsymbol{X},\boldsymbol{A})$, referring to "locally small", resp. "weakly small" anafunctors, are self-explanatory. The latter are full subcategories of ANA$(\boldsymbol{X},\boldsymbol{A})$.

Recall the identification of any functor $G:\boldsymbol{X} \to \boldsymbol{A}$ with an anafunctor; the latter is obviously weakly small. This identification extends to natural transformations, and we have a fully faithful functor $j = j_{\boldsymbol{X},\boldsymbol{A}} : \mathrm{FUN}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A})$, to which we will refer as an *inclusion*.

It is easily seen that if $h: F \to G$ is a natural transformation, then h is an isomorphism in ANA$(\boldsymbol{X},\boldsymbol{A})$ iff each component $h_{s,t}$ is an isomorphism (in $\boldsymbol{A}$) (for h^{-1} defined by $(h^{-1})_{t,s} = (h_{s,t})^{-1}$, we get $h^{-1} \circ h = 1_F$, $h \circ h^{-1} = 1_G$ because the s, s-components of both composites are identities).

Given anafunctors $\boldsymbol{X} \overset{F}{\underset{G}{\rightrightarrows}} \boldsymbol{A}$, *a renaming transformation* $\bar{h}: F \xrightarrow{\equiv} G$ is a system $\bar{h} = \langle h[X,A] \rangle_{X \in \mathrm{Ob}(\boldsymbol{X}),\, A \in \mathrm{Ob}(\boldsymbol{A})}$ of bijections $h[X,A] = (s \mapsto \bar{s}) : |F|(X,A) \xrightarrow{\cong} |G|(X,A)$ preserving the effect of the anafunctors F, G on arrows: $F_{s,s'}(f) = G_{\bar{s},\bar{s}'}(f)$ whenever $f: X \to X'$ is an arrow in $\boldsymbol{X}$, $A \in \boldsymbol{A}$, $s \in |F|(X,A)$, $s' \in |F|(X',A)$. Continuing the above notation,

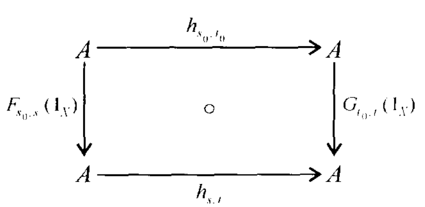
7′. *Every renaming transformation induces a natural isomorphism*

$h: F \xrightarrow{\cong} G$ for which $h_{s,\bar{s}} = 1_A$ $(s \in |F|(X,A))$; condition (3′) holds because of the assumption on effect on arrows (in general, $h_{s,t} = G_{\bar{s},t}(1_X)$ $(s \in |F|X,\ t \in |G|X)$). We now will see that for saturated anafunctors, natural isomorphisms and renaming transformations are in a bijective correspondence.

Suppose that $F, G \in \mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$, and $h: F \xrightarrow{\cong} G$. Let us fix $X \in \boldsymbol{X}$. Note that the isomorphism h in particular ensures that any possible values A, B of F, resp. G, at X are isomorphic; hence, the possible values of F and those of G at X are the same. Let A be any common possible value at X. I claim the following:

8. *For any* $s \in |F|(X,A)$, *there is a unique* $t \in |G|(X,A)$ *such that* $h_{s,t} = 1_A$.

Indeed, let $s_0 \in |F|(X,A)$, $t_0 \in |G|(X,A)$, and consider, with any $s \in |F|(X,A)$ and $t \in |G|(X,A)$, the following commutative diagram of isomorphisms:



This implies that $h_{s,t} = 1_A$ iff

$$\begin{array}{ccc} A & \xrightarrow{h_{s_0,t_0}} & A \\ {\scriptstyle F_{s_0,s}(1_X)}\downarrow & & \downarrow{\scriptstyle G_{t_0,t}(1_X)} \\ A & \xrightarrow[1_A]{} & A \end{array}$$

commutes; the last condition determines $G_{t_0,t}(1_X)$ in terms of (s_0, t_0 and) s; by unique transfer (1(vi)), there is a unique t with this property.

In this argument, we used that G was saturated; using also that F is so, we get

9. *For* $h: F \xrightarrow{\cong} G$ *in* SANA($\boldsymbol{X},\boldsymbol{A}$), $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$, *the condition* $h_{s,\bar{s}} = 1_A$ *for* $s \in |F|(X, A)$, $\bar{s} \in |G|\,(X, A)$ *establishes a bijection* ($s \mapsto \bar{s}$): $|F|\,(X, A) \xrightarrow{\cong} |G|\,(X, A)$ *for which* $F_{s,t}(f) = G_{\bar{s},\bar{t}}(f)$ *holds for all* $f: X \to Y$, $s \in |F|\,X$, $t \in |F|\,Y$.

Therefore by 7′ also, we have the following.

9′. *For* F, $G \in$ ANA($\boldsymbol{X},\boldsymbol{A}$), *the natural isomorphisms* $h: F \xrightarrow{\cong} G$ *are in a bijective correspondence with renaming transformation* $\bar{h}: F \xrightarrow{\equiv} G$.

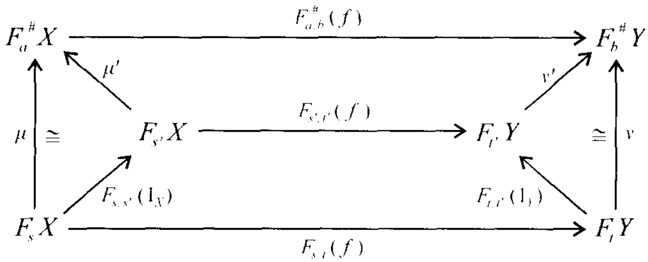
Let us emphasize (in view of the lack of AC) that a functor $\Phi: \boldsymbol{C} \to \boldsymbol{D}$ is an *equivalence of (meta)categories* if there exist a functor $\Psi: \boldsymbol{D} \to \boldsymbol{C}$ and natural isomorphisms $\alpha: 1_{\boldsymbol{C}} \xrightarrow{\cong} \Psi\Phi$, $\beta: 1_{\boldsymbol{D}} \xrightarrow{\cong} \Phi\Psi$. Note that if the functor $\Phi: \boldsymbol{C} \to \boldsymbol{D}$ is full and faithful, and there exists a function $\Psi: \mathrm{Ob}(\boldsymbol{D}) \to \mathrm{Ob}(\boldsymbol{C})$ together with a function $D \mapsto \beta_D$ assigning an isomorphism $\beta_D: D \xrightarrow{\cong} \Phi\Psi D$ to each object $D \in \boldsymbol{D}$ (for which we say that Φ is *uniformly essentially surjective*), then Φ is an equivalence; in fact, there is a unique way of making Ψ into a functor $\Psi: \boldsymbol{D} \to \boldsymbol{C}$ and defining the isomorphism $\alpha: 1_{\boldsymbol{C}} \xrightarrow{\cong} \Psi\Phi$ so that $\langle \beta_D \rangle_D$ becomes an isomorphism $\beta: 1_{\boldsymbol{D}} \xrightarrow{\cong} \Phi\Psi$, and $\alpha\Psi = \Psi\beta$, $\beta\Phi = \Phi\alpha$.

10. *Let* $\boldsymbol{X}$, $\boldsymbol{A}$ *be small categories. The inclusion* $i: \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

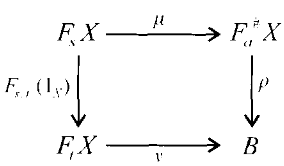
Proof. Let $F \in \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$; we define $F^{\#} \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, called the *saturation* of F, as follows. For $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$, we let $S_{X,A}$ be the set of all pairs ($s \in |F|\,X$, $\mu: F_s X \xrightarrow{\cong} A$). Let $\sim$ be the relation on $S_{X,A}$ defined by

$$(s, \mu) \sim (s', \mu') \quad \underset{\text{def}}{\Longleftrightarrow} \quad \begin{array}{ccc} F_s X & \xrightarrow{F_{s,s'}(1_X)} & F_{s'} X \\ & {\scriptstyle \mu}\searrow\cong \quad \cong\swarrow{\scriptstyle \mu'} & \\ & A & \end{array} \quad \text{commutes.}$$

It is immediately seen that $\sim$ is an equivalence relation. We put $|F^{\#}|\,(X, A)$ to be $S_{X,A}/\sim$, the set of equivalence-classes $[s, \mu]$ of elements (s, μ) of $S_{X,A}$. Given $a = [s, \mu] \in |F^{\#}|\,X$, $b = [t, \nu] \in |F^{\#}|\,Y$ and $f: X \to Y$, $F^{\#}_{a,b}(f)$ is defined so as to make the outside rectangle in the diagram



commute. The commutativity of the rest of the diagram shows that the definition is independent of the choice of the representatives. It is easy to see that 1 (iv), (v) hold for $F^{\#}$ so defined. To see 1(vi) for $F^{\#}$, let $a = [s, \mu] \in |F^{\#}|\,X$, and let $\rho: F^{\#}_a X \xrightarrow{\cong} B$; we want that there is unique $b = [t, \nu]$ with $B = F^{\#}_b X$ and $F^{\#}_{a,b}(1_X) = \rho$; this means that

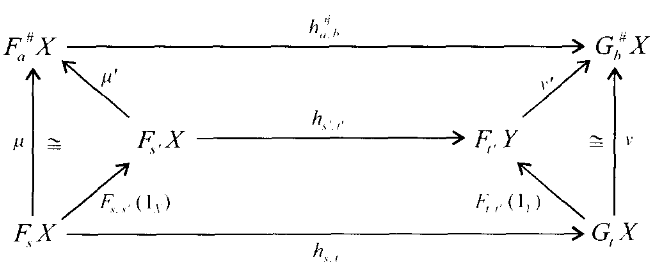


(5)

should commute; we can take $b = [s, \rho \circ \mu]$ to satisfy this; clearly, the commutativity of (5) implies that $(t, \nu) \sim (s, \rho \circ \mu)$, which shows the required uniqueness.

We give $\eta_F: F \xrightarrow{\cong} F^{\#}$ ($F \in$ Ana $(\mathbf{X}, \mathbf{A})$) as an application of 7. We let $I = |F|$, $\langle (s_i, t_i) \rangle_{i \in I} = \langle (s, [s, 1_{\tau(s)}]) \rangle_{s \in |F|}$, with $X_s = \sigma(s)$, and abbreviating $[s, 1_{\tau(s)}]$ as $\bar{s}$, we let, for $s \in |F|\,X$, $(\eta_F)_s: F_s(X) \to F^{\#}_{\bar{s}}(X)$ be the identity $1_{F_s X}$. It is immediate that η_F is a natural transformation (by 7; (3′) now holds), and that it is an isomorphism. This completes the proof of 10.

Let us note the effect of the saturation functor $(\ )^{\#}: \mathrm{Ana}(\mathbf{X},\mathbf{A}) \to \mathrm{Sana}(\mathbf{X},\mathbf{A})$ on arrows. Given $h: F \to G$ in $\mathrm{Ana}(\mathbf{X},\mathbf{A})$, $a = [s, \mu] \in F^{\#}\,X$, $b = [t, \nu] \in G^{\#}\,X$, $h^{\#}_{a,b}$ is defined so as to make the outside rectangle in



commute; the rest of the diagram shows that the definition of $h^{\#}_{a,b}$ is independent of the choice of the representatives; it is easy to see that $h^{\#}$ so defined is a natural transformation $F^{\#} \to G^{\#}$. Further, it is easily seen that $(\ )^{\#}$ so defined is a functor. The functor $(\ )^{\#}$ is the same as the one obtained from i and $\langle \eta_F \rangle_F$ in the remark before 10, denoted by Ψ there.

Given a weakly small anafunctor $F: \mathbf{X} \to \mathbf{A}$, *using* the Global Axiom of Choice (GAC), the existence of a class-function that picks an element of every inhabited set, we let $(X \in \mathbf{X}) \mapsto s_X \in |F| X$ be a choice-function, and we consider the functor $F^{!}: \mathbf{X} \to \mathbf{A}$ for which $F^{!}(X) = F_{s_X}(X)$, $F^{!}(f: X \to Y) \underset{\text{def}}{=} F_{s_X, s_Y}(f): F^{!}_X \to F^{!} Y$ (it is immediate that $F^{!}$ is a functor). We also have, with any F as above, a natural isomorphism $\alpha: F \to jF^{!}$ (with j the inclusion of functors in anafunctors) defined by

$$(\alpha_{s,X}: F_s X \to F_{s_X} X) \underset{\text{def}}{=} (F_{s,s_X}(1_X): F_s X \to F_{s_X} X).$$

Making the choices involved simultaneously for all $F \in \mathrm{ANA}_{\mathrm{ws}}(\mathbf{X}, \mathbf{A})$, we obtain, *using* the GAC, that

11. (GAC). *The inclusion* $\mathrm{FUN}(\mathbf{X}, \mathbf{A}) \to \mathrm{ANA}_{\mathrm{ws}}(\mathbf{X}, \mathbf{A})$ *is an equivalence of meta-categories; when* $\mathbf{X}$ *is small, the inclusion* $\mathrm{Fun}(\mathbf{X}, \mathbf{A}) \to \mathrm{Ana}(\mathbf{X}, \mathbf{A})$ *is an equivalence of categories.*

11 is reassuring since it says that we have not strayed from the notion of functor too far.

It should be noted that, without any choice,

11′. *Any small anafunctor into* Set *is isomorphic to a functor; for any small category* $\mathbf{X}$, *the inclusion* $\mathrm{Fun}(\mathbf{X}, \mathrm{Set}) \to \mathrm{Ana}(\mathbf{X}, \mathrm{Set})$ *is an equivalence of categories.*

Proof. Let $F: \mathbf{X} \to \mathrm{Set}$ be a small anafunctor. An *element of F at $X \in \mathbf{X}$* is a family $x = \langle x_s \rangle_{s \in |F| X}$ such that $x_s \in F_s X$, and $(F_{s,t}(1_X))(x_s) = x_t$ for $s, t \in |F| X$. Clearly, any component x_s of x determines the whole of x, and in fact, any pair $(s \in |F| X, a \in F_s X)$ determines a unique element x at X for which $x_s = a$; let us denote x by $[s, a]$.

Given F, we define the functor $\hat{F}: \mathbf{X} \to \mathrm{Set}$ as follows. We put $\hat{F}(X)$ equal to the set of all elements of F at X. We define, for $f: X \to Y$, the function $\hat{F}(f): \hat{F}(X) \to \hat{F}(Y)$ by putting $\hat{F}(f)(x)$ equal to the unique element y at Y for which $y_u = F_{s,u}(f)(x_s)$ for any (equivalently, for some) pair $(s \in |F| X,\ u \in |F| Y)$. It is easily seen that $\hat{F}$ is well-defined as a functor $\hat{F}: \mathbf{X} \to \mathrm{Set}$ by these stipulations. We have the natural isomorphism $\alpha_F: F \xrightarrow{\cong} \hat{F}$ whose components $(\alpha_F)_{s,X}: F_s X \to \hat{F} X$ are given by $(\alpha_F)_{s,X}(a) = [s, a]$. This completes the proof.

Many concrete categories (categories of algebras, of topological spaces, etc.) that have a faithful forgetful functor to Set share the property of Set stated in 11′; I do not see how to make a general-enough statement of this state of affairs.

Another, rather obvious, case of this situation is in the next statement.

11″. *The anafunctor F is isomorphic to an ordinary functor when the domain category of F has finitely many objects.*

(By "S is finite", we mean "there are a natural number n and a surjection $\{i : i < n\} \to S$".) Note, however, that we cannot say that the inclusion $\mathrm{Fun}(\boldsymbol{X}, \boldsymbol{A}) \to \mathrm{Ana}(\boldsymbol{X}, \boldsymbol{A})$ is an equivalence even when $\boldsymbol{X}$ is $\mathbf{1}$, the terminal category.

We turn to the composition of anafunctors. Let $\boldsymbol{X} \xrightarrow{F} \boldsymbol{A} \xrightarrow{G} \boldsymbol{M}$ be anafunctors. There is a natural composition $G \circ F : \boldsymbol{X} \to \boldsymbol{A}$, also written just GF, an anafunctor, defined as follows. For $X \in \boldsymbol{X}$, we let $|GF|\,X$ be the class of all pairs

$$a = (s \in |F|\,X, t \in |G|(F_s X))$$

(in other words,

$$|GF|\,X \underset{\mathrm{def}}{=} \bigsqcup_{s \in |F|\,X} |G|(F_s X)) \tag{6}$$

and for a as displayed, $(GF)_a \underset{\mathrm{def}}{=} G_t(F_s(X))$. Note that if also $M \in \boldsymbol{M}$,

$$|GF|(X, M) \cong \bigsqcup_{A \in \boldsymbol{A}} |F|(X, A) \times |G|(A, M).$$

For the action of GF on arrows, with a as above, and with $b = (u \in |F|\,Y, v \in |G|(F_u X))$, and with $f : X \to Y$, we put $(GF)_{a,b}(f) \underset{\mathrm{def}}{=} G_{t,v}(F_{s,u}(f))$. It is immediate that GF is an anafunctor.

It is immediate that the composition of weakly small anafunctors is weakly small. If F and G are given by the spans $(\boldsymbol{X} \leftarrow |F| \to \boldsymbol{A})$, $(\boldsymbol{A} \leftarrow |G| \to \boldsymbol{M})$, then the composite GF is given by the "composite span" $(\boldsymbol{X} \leftarrow |F| \times_{\boldsymbol{A}} |G| \to \boldsymbol{M})$.

We can extend composition to a functor

$$\mathrm{ANA}(\boldsymbol{X}, \boldsymbol{A}) \times \mathrm{ANA}(\boldsymbol{A}, \boldsymbol{M}) \longrightarrow \mathrm{ANA}(\boldsymbol{X}, \boldsymbol{M}) \tag{7}$$

in a natural way. With data as in

$$\boldsymbol{X} \underset{G}{\overset{F}{\underset{\Downarrow h}{\rightrightarrows}}} \boldsymbol{A} \underset{J}{\overset{I}{\underset{\Downarrow k}{\rightrightarrows}}} \boldsymbol{M} \tag{8}$$

first we define $\circ(h, 1_I)$, denoted Ih, by

$$(Ih)_{a,b} = I_{t,v}(h_{s,u});$$

here, $a = (s \in |F|\,X, t \in |I|(F_s X))$, $b = (u \in |G|\,X, v \in |I|(F_u X))$; the naturality of Ih is immediate.

In defining $\circ(1_F, k)$, denoted kF, we make use of the fact that, to specify a natural transformation of anafunctors, it suffices to specify "enough" components of it, with the appropriate naturality conditions satisfied (see 7). Accordingly, let $a = (s \in |F|\,X, t \in |I|(F_s X))$, $b = (s \in |F|\,X, u \in |J|(F_s X))$; we let $(kF)_{a,b} : (IF)_a(X) \to (JF)_b(X)$ be $(kF)_{a,b} \underset{\mathrm{def}}{=} k_{t,u} : I_t F_s(X) \to J_u F_s(X)$; the (needed partial) naturality of kF is immediate.

Next, we need to verify that thus we have defined functors

$$(\ )\circ I : \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{M}),$$

$$F\circ(\ ) : \mathrm{ANA}(\boldsymbol{A},\boldsymbol{M}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{M});$$

we leave the task to the reader.

Finally, we need that

$$\begin{array}{ccc} I\circ F & \xrightarrow{\ Ih\ } & I\circ G \\ {\scriptstyle kF}\downarrow & & \downarrow{\scriptstyle kG} \\ J\circ F & \xrightarrow[\ Jh\ ]{} & J\circ G \end{array}$$

commutes. With evaluating $I\circ F$ at $(s\in |F|X,\ t\in |I|(F_sX))$, $J\circ G$ at $(u\in |G|X,$ $v\in |J|(G_uX))$, $I\circ G$ at $(u\in |G|X,\ w\in |I|(G_uX))$, and $J\circ F$ at $(s\in |F|X,\ r\in |J|(F_sX))$, the diagram becomes

$$\begin{array}{ccc} I_tF_sX & \xrightarrow{\ I_{t,w}(h_{s,u})\ } & I_wG_uX \\ {\scriptstyle k_{t,r}}\downarrow & & \downarrow{\scriptstyle k_{w,v}} \\ J_rF_sX & \xrightarrow[\ J_{r,v}(h_{s,u})\ ]{} & J_vG_uX \end{array}$$

whose commutativity is an instance of the naturality of k. By 7 again, this suffices.

It is well known (Proposition 1, II.3, p. 37 in [11]) that what we did above determines uniquely the functor (7).

It is clear that, for $\boldsymbol{X}$ and $\boldsymbol{A}$ small, (7) restricts to a composition-functor

$$\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})\times \mathrm{Ana}(\boldsymbol{A},\boldsymbol{M}) \xrightarrow{\ \circ\ } \mathrm{Ana}(\boldsymbol{X},\boldsymbol{M}). \tag{7'}$$

Let us turn to the question of associativity of composition of functors. With anafunctors

$$\boldsymbol{X}\xrightarrow{\ F\ }\boldsymbol{A}\xrightarrow{\ G\ }\boldsymbol{M}\xrightarrow{\ H\ }\boldsymbol{S},$$

we find the associativity isomorphism

$$\alpha = \alpha_{F,G,H} : H(GF) \xrightarrow{\ \cong\ } (HG)F$$

given (see 7′) by the renaming transformation $\bar{\alpha}$ for which

$$\alpha[X,S] : ((s,t),u) \mapsto (s,(t,u))$$

whenever $X\in\boldsymbol{X}$, $s\in|F|X$, $t\in|G|(F_sX)$, $u\in|H|(G_tF_s(X))$, $S=H_uG_tF_s(X)$. It is easy to see that $\alpha_{F,G,H}$ is natural in each of F, G and H, and that the pentagonal associativity coherence diagram ((1.1) (A.C.) in [1], pp. 5 and 6) commutes. With the identity functor $1_A : A \to A$ as an anafunctor, we have the left and right identity isomorphisms

$$\lambda_F : 1_A F \xrightarrow{\cong} F, \qquad \rho_F : I1_A \xrightarrow{\cong} I$$

(see (8)) defined by $(\lambda_F)_{((s,X),s),X} = 1_{F_s X}$ $(s \in |F|\, X)$, and similarly for ρ_F. Both λ_F and ρ_F are natural in F, and they satisfy identity coherence ((1.1) (I.C.) *loc.cit.*).

We have the ingredients of a metabicategory (see *loc. cit.*).

12. Conclusion. *Categories, anafunctors between them, and natural transformations between the latter form, with the given notions of composition, a meta-bicategory* ANACAT. *The identification of ordinary functors with anafunctors provides an inclusion* i: CAT → ANACAT (CAT *is the meta-2-category of categories, functors and natural transformations*), *which is the identity on objects, and locally fully faithful.*

We also have the bicategory AnaCat *of small categories, small anafunctors between them, and all natural transformations between the latter. The 2-category* Cat *of small categories has a locally fully faithful inclusion into* AnaCat, *which is an equivalence of bicategories provided the Axiom of Choice holds.*

G.M. Kelly gave us once the healthy advice to use simple terminology in higher dimensional category theory. For instance, "functor" of bicategories should mean "homomorphism of bicategories"; a functor between bicategories cannot reasonably mean anything but a mapping that respects the whole bicategory structure and not just the reduct to the category structure. Similarly, "product" in a bicategory should mean what is usually called "*bi*product". Also, I say "equivalence of bicategories" for "biequivalence". (As a reminder, I note that by an *equivalence of bicategories* $\mathscr{S}$ and $\mathscr{A}$, I mean a pair of functors $\mathscr{S} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{A}$ such that $GF \simeq 1_{\mathscr{S}}$, $FG \simeq 1_{\mathscr{A}}$, the latter equivalences meant in the metabicategories of endofunctors of $\mathscr{S}$, $\mathscr{A}$, respectively. As usual, we say of a single functor $F : \mathscr{S} \to \mathscr{A}$ that it is an equivalence if it can be expanded with further data to form an equivalence.) Maybe I am carrying Kelly's advice farther than he intended; I hope no confusion will arise.

Small categories with saturated anafunctors between them also form a bicategory named SanaCat, which is equivalent to AnaCat. This is a consequence of 10, together with the fact that, in the proof of 10, the isomorphisms η_F are obtained uniformly from F not just within a given Ana$(\boldsymbol{X}, \boldsymbol{A})$, but also uniformly in the variables $\boldsymbol{X}$, $\boldsymbol{A}$.

In some detail, SanaCat has the following structure. With reference to the saturation-functor

$$(\ )^{\#} = (\ )^{\#}_{X,M} : \mathrm{Ana}(\boldsymbol{X}, \boldsymbol{M}) \to \mathrm{Sana}(\boldsymbol{X}, \boldsymbol{M})$$

(see 10), a composition-functor in SanaCat,

$$\circ^{\#} = \circ^{\#}_{X,A,M} : \mathrm{Sana}(\boldsymbol{X}, \boldsymbol{A}) \times \mathrm{Sana}(\boldsymbol{A}, \boldsymbol{M}) \to \mathrm{Sana}(\boldsymbol{X}, \boldsymbol{M}),$$

is defined by $G \circ^{\#} F = (G \circ F)^{\#}$, and correspondingly for natural transformations. The associativity isomorphisms

$$\alpha^{\#}_{F,G,H} : H \circ^{\#} (G \circ^{\#} F) \to (H \circ^{\#} G) \circ^{\#} F$$

are determined so as to make

$$\begin{array}{ccc} H\circ^{\#}(G\circ^{\#}F) & \xrightarrow{\alpha^{\#}_{F,G,H}} & (H\circ^{\#}G)\circ^{\#}F \\ \uparrow{\scriptstyle\eta} & & \uparrow{\scriptstyle\eta} \\ H\circ(G\circ^{\#}F) & & (H\circ^{\#}G)\circ F \\ \uparrow{\scriptstyle H\cdot\eta} & & \uparrow{\scriptstyle\eta\cdot F} \\ H\circ(G\circ F) & \xrightarrow[\alpha_{F,G,H}]{} & (H\circ G)\circ F \end{array}$$

commute.

Using 10, we can see that

12′. *The inclusion mapping* SanaCat $\to$ AnaCat *is an equivalence of bicategories.*

It is more natural to make the totality of small categories, with saturated anafunctors between them, an *anabicategory* in which composition is an anafunctor; see Section 4.

Terminal object and *product* in a (meta-)bicategory are defined as expected by universal properties defining the result of the operation up to an equivalence rather than isomorphism. Placing ourselves in a fixed (meta)bicategory, we say that $A \xleftarrow{\pi} C \xrightarrow{\pi'} B$ is a *product diagram* if, for any object D, the functor

$$(\pi(-), \pi'(-)) : \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, A) \times \mathrm{Hom}(D, B)$$

$$D \xrightarrow{f} C \mapsto (\pi f, \pi' f)$$

is an equivalence of categories. As usual, $A \xleftarrow{\pi} A \times B \xrightarrow{\pi'} B$ denotes, ambiguously, a product diagram on (A, B).

T is a *terminal object* if, for any A, $\mathrm{Hom}(A, T) \to \mathbf{1}$, with $\mathbf{1}$ the one-object, one-arrow category, is an equivalence of categories.

We say that a bicategory is *Cartesian* if it has a terminal object and binary products.

13. AnaCat *and* ANACAT *are Cartesian*

In fact, the Cartesian structure in ANACAT (AnaCat) is computed as in CAT (Cat).

The Cartesian closed nature of Cat, the (2-)category of all small categories is a fundamental fact. What prevents AnaCat from being Cartesian closed is that, for $\boldsymbol{A}$, $\boldsymbol{X}$ small categories, Ana$(\boldsymbol{X}, \boldsymbol{A})$ is not neccessarily equivalent to a small category. In

Section 5, we will see that a weak form of the AC will ensure this, and hence the Cartesian closed nature of AnaCat. Here, we give the relevant facts that hold without further set-theoretical hypotheses.

We first formulate a characterization of anafunctors of the form $F:\boldsymbol{X}\times\boldsymbol{M}\to\boldsymbol{A}$, ("bi-anafunctors") analogous to Proposition II.3.1 in [11]. Suppose we have

classes $|F|\,((X,M))\quad (X\in\boldsymbol{X},\ M\in\boldsymbol{M})$,

objects $F_s(X,M)\in\boldsymbol{A}\quad (s\in|F|\,((X,M)))$,

arrows $F_{s,t}(f,M):F_s(X,M)\to F_t(Y,M)$,

$F_{s,u}(X,g):F_s(X,M)\to F_u(X,N)\quad (f:X\to Y,\ g:M\to N,\ s\in|F|\,((X,M))$,

$t\in|F|\,((Y,M)),\ u\in|F|\,((X,N)))$

such that

(i) for any $X\in\boldsymbol{X}$, the data define an anafunctor $F_X=F(X,-):\boldsymbol{M}\to\boldsymbol{A}$ ($|F_X|M=|F|\,((X,M))$, etc.), and similarly for $F(-,M):\boldsymbol{X}\to\boldsymbol{A}$;

(ii) for any $f:X\to Y$ in $\boldsymbol{X}$, $g:M\to N$ in $\boldsymbol{M}$, and for all appropriate specifications, the diagram

$$\begin{array}{ccc} F_s(X,M) & \xrightarrow{F_{s,t}(X,g)} & F_t(X,N) \\ {\scriptstyle F_{s,u}(f,M)}\downarrow & & \downarrow{\scriptstyle F_{t,v}(f,N)} \\ F_u(Y,M) & \xrightarrow[F_{u,v}(Y,g)]{} & F_v(Y,N) \end{array}$$

commutes. Then we have a unique anafunctor $F:\boldsymbol{X}\times\boldsymbol{M}\to\boldsymbol{A}$ having as sections $F(X,-)$, $F(-,M)$ the given data.

I leave the verification to the reader.

Given categories $\boldsymbol{X},\boldsymbol{A}$, we consider the metacategory ANA $(\boldsymbol{X},\boldsymbol{A})$, and the *evaluation* anafunctor

$$e=e_{\boldsymbol{X},\boldsymbol{A}}:\boldsymbol{X}\times\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})\to\boldsymbol{A} \tag{8}$$

determined as follows. For $X\in\boldsymbol{X}$, $F\in\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$,

$$|e|\,((X,F))\underset{\mathrm{def}}{=}|F|\,X; \tag{8'}$$

for $s\in|F|\,X$,

$$e_s(X,F)\underset{\mathrm{def}}{=}F_s(X);$$

with also $u\in|F|\,Y, f:X\to Y$,

$$e_{s,u}((f,F))\underset{\mathrm{def}}{=}F_{s,u}(f).$$

With $h: F \to G$ ($\in$ Arr (ANA($\boldsymbol{X},\boldsymbol{A}$))), $t \in |G|X$,

$$e_{s,t}((X,h)) \underset{\text{def}}{=} h_{s,t};$$

the diagram

$$\begin{array}{ccc}
e_s(X,F) & \xrightarrow{e_{s,t}(X,h)} & e_t(X,G) \\
{\scriptstyle e_{s,u}(f,F)}\big\downarrow & & \big\downarrow{\scriptstyle e_{t,v}(f,G)} \\
e_u(Y,F) & \xrightarrow[e_{u,v}(Y,h)]{} & e_v(Y,G)
\end{array}$$

is identical to

$$\begin{array}{ccc}
F_s(X) & \xrightarrow{h_{s,t}} & G_t(X) \\
{\scriptstyle F_{s,u}(f)}\big\downarrow & & \big\downarrow{\scriptstyle G_{t,v}(f)} \\
F_u(Y) & \xrightarrow[h_{u,v}]{} & G_v(Y)
\end{array}$$

which commutes by the naturality of h. This shows (by the characterization of "bi-anafunctors") that e is an anafunctor.

Whereas $e_{X,A}$ in (8) is a metafunctor, for $\boldsymbol{X}$ small, its restriction

$$e_{X,A}: \boldsymbol{X} \times \text{Ana}(\boldsymbol{X},\boldsymbol{A}) \to \boldsymbol{A} \tag{9}$$

to Ana($\boldsymbol{X},\boldsymbol{A}$), the category of small anafunctors $\boldsymbol{X} \to \boldsymbol{A}$, is a functor (denoted by the same symbol as the metafunctor in (8)).

In propositions 14, 15, 16 and 17 below, $\boldsymbol{X}$, $\boldsymbol{Y}$ are small categories, $\boldsymbol{A}$ is an arbitrary category.

14. $e = e_{X,A}$ (see (9)) *induces an equivalence of categories*

$$\varphi \underset{\text{def}}{=} e \circ (\boldsymbol{X} \times (-)) : \text{Ana}(\boldsymbol{Y}, \text{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\simeq} \text{Ana}(\boldsymbol{X} \times \boldsymbol{Y},\boldsymbol{A}).$$

15. *The inclusion*

$$i: \text{Fun}(\boldsymbol{Y}, \text{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\simeq} \text{Ana}(\boldsymbol{Y}, \text{Ana}(\boldsymbol{X},\boldsymbol{A}))$$

is an equivalence of categories.

Note that 15 implies that Ana($\boldsymbol{X},\boldsymbol{A}$) shares the property of Set given in 11′.

16. *There is an isomorphism*

$$\psi : \text{Fun}(\boldsymbol{Y}, \text{Ana}(\boldsymbol{X},\boldsymbol{A})) \xrightarrow{\cong} \text{Ana}(\boldsymbol{X} \times \boldsymbol{Y},\boldsymbol{A})$$

of categories for which $\psi \cong i \circ \varphi$, *with* i *and* φ *from* 15 *and* 14.

Proof of 14, 15 and 16. The functors in these assertions form the diagram

$$\begin{array}{ccc} \mathrm{Ana}\,(\boldsymbol{Y}, \mathrm{Ana}\,(\boldsymbol{X},\boldsymbol{A})) & \xrightarrow{\quad\varphi\quad} & \mathrm{Ana}\,(\boldsymbol{X}\times\boldsymbol{Y},\boldsymbol{A}) \\ {\scriptstyle i}\nwarrow & & \nearrow{\scriptstyle \psi} \\ & \mathrm{Fun}\,(\boldsymbol{Y}, \mathrm{Ana}\,(\boldsymbol{X},\boldsymbol{A})) & \end{array} \tag{9}$$

We will define ψ, show the properties given in 16, and show that φ is full and faithful. Since i is full and faithful, both assertions 14 and 15. will follow. We will have that, in (9), all three functors are equivalences of categories, one in fact is an isomorphism.

Given $H \in \mathrm{Ana}(\boldsymbol{Y}, \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$, $X \in \boldsymbol{X}$, $Y \in \boldsymbol{Y}$, we have

$$|e\circ(\boldsymbol{X}\times H)|\,((X,Y)) = \{((X,a),s) : a\in|H|\,Y,\ s\in|H_a\,Y\,|\,X\}$$

(remember that $|e|\,((X,H_a\,Y)) = |H_a\,Y\,|\,X)$ and

$$(e\circ(\boldsymbol{X}\times H))_{((X,a),s)}(X,Y) = (H_a\,Y)_s X.$$

Let also $K \in \mathrm{Ana}(\boldsymbol{Y}, \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$. A natural transformation $h : e\circ(\boldsymbol{X}\times H) \to e\circ(\boldsymbol{X}\times K)$ has components

$$h_{((X,a),s),\ ((X,b),t)} : (H_a\,Y)_s X \to (K_b\,Y)_t X.$$

Starting with h, we define $j : H \to K$ by specifying the natural transformation $j_{a,b} : H_a\,Y \to K_b\,Y$ by making $(j_{a,b})_{s,t} : (H_a\,Y)_s X \to (K_b\,Y)_t X$ equal to $h_{((X,a),s),\ ((X,b),t)}$. This works, and j is the unique natural transformation $H \to K$ mapped by the functor (9) to h; this amounts to the fact that φ is fully faithful.

Given the small anafunctor $G : \boldsymbol{X}\times\boldsymbol{Y} \to \boldsymbol{A}$, we define $H = \psi^{-1}\,(G)$, $H : \boldsymbol{Y} \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ as follows. With $Y \in \boldsymbol{Y}$, $H\,(Y) : \boldsymbol{X} \to \boldsymbol{A}$ is the (obviously small) anafunctor $G\,(-\,,\,Y)$, that is

$$|H(Y)|\,X = |G|\,((X,Y)), \qquad (H(Y))_s X = G_s(X,Y)$$

and

$$(H(Y))_{s,t}(f) = G_{s,t}(f,Y) \quad (s\in|H(Y)|\,X,\ t\in|H(Y)|\,X',\, f : X \to X');$$

moreover, for $g : Y \to Y'$, $H(g) : H(Y) \to H(Y')$ is the natural transformation for which

$$(H(g))_{s,t} = G_{s,t}(X,g).$$

Conversely, given any functor $H : \boldsymbol{Y} \to \mathrm{Ana}\,(\boldsymbol{X},\boldsymbol{A})$, the listed equalities define a unique small $G : \boldsymbol{X}\times\boldsymbol{Y} \xrightarrow{\ \mathrm{a}\ } \boldsymbol{A}$; in other words, ψ is a bijection of the object-classes of the two categories in 16. If $g : G \to F$, then $\psi^{-1}(g) = h : H \to K$ for h defined by

$$(h_Y)_{s,t} = g_{s,t} \quad (s\in|G|\,(X,Y),\ t\in|F|\,(X,Y))$$

(here, G, $F \in \mathrm{Ana}(\boldsymbol{X} \times \boldsymbol{Y}, \boldsymbol{A})$, $H = \psi^{-1}(G)$, $K = \psi^{-1}(F)$), and the mapping $g \mapsto h$ is a bijection $\mathrm{Nat}(G,F) \xrightarrow{\cong} \mathrm{Nat}(\psi^{-1}G, \psi^{-1}F)$. This defines the isomorphism ψ of 16.

To show the isomorphism $\psi \cong i \circ \varphi$, for a functor $H: \boldsymbol{Y} \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, and $G = \psi(H)$, we exhibit an isomorphism $\alpha_H : e \circ (\boldsymbol{X} \times H) \cong G$. Calculating $e \circ (\boldsymbol{X} \times H)$ in this case, we get

$$|e \circ (\boldsymbol{X} \times H)|((X, Y), A) = \{((X, Y), s) : s \in |G|\,((X, Y),\ A)\}.$$

We can define the renaming transformation $\bar{\alpha} : e \circ (\boldsymbol{X} \times H) \xrightarrow{\equiv} G$ by defining

$$\bar{\alpha}[(X, Y), A] : |e \circ (\boldsymbol{X} \times H)|((X, Y), A) \xrightarrow{\cong} |G|\,((X, Y), A)$$

as

$$((X, Y), s) \to s.$$

The corresponding natural isomorphism $\alpha_H : e \circ (\boldsymbol{X} \times H) \xrightarrow{\cong} G$ has

$$(\alpha_H)_{((X,Y),s),s} = 1_{G_s(X,Y)}. \tag{10}$$

We need to see that α_H is natural in $H \in \mathrm{Fun}(\boldsymbol{Y}, \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$. Because of (10), naturality means that for H, $K \in \mathrm{Fun}(\boldsymbol{Y}, \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$, $j : H \to K$, $h = \varphi(j)$, $\ell = \psi^{-1}(j)$, $s \in |H(Y)|\,X$, $t \in |K(Y)|\,X$, we have

$$h_{((X,Y),s),\,((X,Y),t)} = \ell_{s,t}(:(HY)_s X \to (KY)_t X).$$

But this equality is true; both sides are equal to $(j_Y)_{s,t}$.

This completes the proof.

We also arrive at the conclusion mentioned after 15: If $K : \boldsymbol{Y} \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ is an anafunctor, we have a functor $H : \boldsymbol{Y} \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ isomorphic to it; H is obtained from $G = e \circ (\boldsymbol{X} \times K)$ as above. In particular, the anafunctor $H(Y) : \boldsymbol{X} \to \boldsymbol{A}$ has

$$|H(Y)|\,X = \{(a,s) : a \in |K|\,Y,\ s \in |K_a(Y)|\,X\};$$

the "uncertainty" from K is absorbed into the values of H.

Here is a rather special, but useful, result.

17. *When the category* $\boldsymbol{X}$ *has finitely many objects, the functor*

$$\imath \circ (\ ) : \mathrm{Ana}(\boldsymbol{Y}, \mathrm{Fun}(\boldsymbol{X},\boldsymbol{A})) \to \mathrm{Ana}(\boldsymbol{Y}, \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}))$$

induced by the inclusion $\imath : \mathrm{Fun}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

Proof. Since $\imath$ is full and faithful, it is immediate that so is $\imath \circ (\ )$. To show that $\imath \circ (\ )$ is uniformly essentially surjective on objects, it suffices to show that the composite with the equivalence φ of (9),

$$\varphi \circ (\imath \circ (\ )) : \mathrm{Ana}(\boldsymbol{Y}, \mathrm{Fun}(\boldsymbol{X},\boldsymbol{A})) \to \mathrm{Ana}(\boldsymbol{X} \times \boldsymbol{Y},\boldsymbol{A}),$$

is so. Let $G:\boldsymbol{X}\times\boldsymbol{Y}\xrightarrow{a}\boldsymbol{A}$. Define $F:\boldsymbol{Y}\xrightarrow{a}\mathrm{Fun}\,(\boldsymbol{X},\boldsymbol{A})$ as follows. Put $|F|\,Y \underset{\mathrm{def}}{=} \prod_{X\in|\boldsymbol{X}|}|G|\,((X,Y))$; for $a\in|F|\,Y$, $F_a(Y)(X)\underset{\mathrm{def}}{=}G_{a(X)}(X,Y)$; for $x:X\to X'$, $F_a(Y)(x)\underset{\mathrm{def}}{=}G_{a(X),a(X')}\,(x,Y)$ (note that $a(X)\in|G|\,((X,Y))$, $a(X')\in|G|\,((X',Y))$); for $y:Y\to Y'$, $a\in|F|\,Y$, $a'\in|F|\,Y'$, the components of the natural transformation $F_{a,a'}(f):F_a\,(Y)\to F_{a'}\,(Y')$ are defined as

$$(F_{a,a'}(f))_X \underset{\mathrm{def}}{=} G_{a(X),a'(X)}(X,y):G_{a(X)}(X,Y)\to G_{a'(X)}(X,Y').$$

It is easy to check that F is an anafunctor; the only point where the finiteness of $|\boldsymbol{X}|$ is used is the inhabitedness of the set $|F|\,Y=\prod_{X\in|\boldsymbol{X}|}|G|\,((X,Y))$; as a finite product of inhabited sets, it is inhabited.

We need to exhibit a natural isomorphism $h:(\varphi\circ(\iota\circ(\ ))\,(F)\xrightarrow{\cong}G$. But $(\varphi\circ(\iota\circ(\ ))\,(F)=e\circ(\boldsymbol{X}\times\iota(F))$ has

$$|e\circ(\boldsymbol{X}\times\iota(F))|\,((X,Y))=\{((X,a),X):a\in|F|\,Y\}$$

and

$$(e\circ(\boldsymbol{X}\times\iota(F)))_{((X,a),X)}(X,Y)=F_a(Y)(X)(=G_{a(X)}(X,Y)).$$

Thus, we may define h by

$$h_{((X,a),X),a(X)}=1_{G_{a(X)}(X,Y)};$$

7 ensures that h is well-defined.

When in 16, we put $\boldsymbol{X}=\mathbf{1}$, we note the isomorphism $\mathbf{1}\times\boldsymbol{Y}\cong\boldsymbol{Y}$, and we write $\boldsymbol{A}^+$ for Ana $(\mathbf{1},\boldsymbol{A})$ (we may call $\boldsymbol{A}^+$ the category of *small anaobjects* of $\boldsymbol{A}$), we obtain the isomorphism $\mathrm{Ana}(\boldsymbol{Y},\boldsymbol{A})\cong\mathrm{Fun}(\boldsymbol{Y},\boldsymbol{A}^+)$ of categories. In other words, (small) anafunctors $\boldsymbol{Y}\to\boldsymbol{A}$ may be identified with ordinary functors from the same domain $\boldsymbol{Y}$ into the category $\boldsymbol{A}^+$ of (small) anaobjects of the codomain $\boldsymbol{A}$, and this identification extends to natural transformations. This shows that the notion of anafunctor and that of natural transformation of anafunctors can be reduced to the case when the domain category is $\mathbf{1}$. This fact was suggested by the referee.

When in 14, we put both $\boldsymbol{X}$ and $\boldsymbol{Y}$ equal to $\mathbf{1}$, we obtain the equivalence $\boldsymbol{A}^{++}\simeq\boldsymbol{A}^+$. In fact, writing $\mu_A:\boldsymbol{A}^{++}\xrightarrow{\simeq}\boldsymbol{A}^+$ for a (the) quasi-inverse of the equivalence $\varphi:\boldsymbol{A}^+\xrightarrow{\cong}\boldsymbol{A}^{++}$ given in 14, and $\eta_A:\boldsymbol{A}\to\boldsymbol{A}^+$ for the inclusion functor $\boldsymbol{A}\cong\mathrm{Fun}\,(\mathbf{1},\boldsymbol{A})\to\mathrm{Ana}(\mathbf{1},\boldsymbol{A})$, we have an idempotent monad $((\ )^+,\mu,\eta)$ on the bicategory AnaCat (both "idempotent" and "monad" understood in the suitable bicategorical sense); this fact will be explored in [15]. Further, in [15], it will be shown that $\boldsymbol{A}^+$ is a *stack-completion* of $\boldsymbol{A}$; the full explanation of this fact requires putting anafunctors into the context of indexed category theory.

As I mentioned in the Introduction, the construction of the category $\boldsymbol{A}^+$ is also given in [10], where $\boldsymbol{A}^+$ is named the category of *cliques* of $\boldsymbol{A}$; see [10, Chapter 1, Section 1]. The general properties of cliques and $\boldsymbol{A}^+$ are not developed in [10]; $\boldsymbol{A}^+$ is used in [10] for purposes different from those of this paper.

Written out explicitly, $\boldsymbol{A}^+$ is the following category. An object A of $\boldsymbol{A}^+$ (a clique, or a small anaobject of $\boldsymbol{A}$) is given by an inhabited set $|A|$, an $|A|$-indexed family $\langle A_s\rangle_{s\in|A|}$

of objects A_s of $\boldsymbol{A}$, and an assignment of an isomorphism $A_{s,t}: A_s \xrightarrow{\cong} A_t$ to each pair (s, t) of elements of S such that $A_{s,s} = 1_{A_s}$ and $A_{t,u} \circ A_{s,t} = A_{s,u}$ whenever s, t, $u \in S$. A morphism $h: A \to A'$ is a family

$$h = \langle h_{s,s'}: A_s \to A'_{s'} \rangle_{s \in |A|, s' \in |A'|}$$

such that

$$\begin{array}{ccc} A_s & \xrightarrow{h_{s,s'}} & A_{s'} \\ {\scriptstyle A_{s,t}}\downarrow & \circ & \downarrow{\scriptstyle A_{s',t'}} \\ A_t & \xrightarrow[h_{t,t'}]{} & A_{t'} \end{array}$$

for all appropriate values of the parameters.

By a (not necessarily small) anaobject of $\boldsymbol{A}$, we mean a (not necessarily small) anafunctor $\mathbf{1} \to \boldsymbol{A}$; we will use (in Section 3) a similar notation in relation to anaobjects in general as we did above for small anaobjects; for a general anaobject A, $|A|$ may be proper class.

2. Adjoint anafunctors

Anafunctors provide solutions without introducing non-canonical choices to existence problems when data are given by universal properties. The best example for this is the existence of an adjoint anafunctor when the "local existence criterion" is satisfied.

Given the anafunctors $\boldsymbol{X} \underset{G}{\overset{F}{\rightleftarrows}} \boldsymbol{A}$, we say that F is a *left-adjoint to* G ($F \dashv G$) if we have, for any $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$, $s \in |F| X$, $v \in |G| A$ a bijection $\varphi_{s,v}$, mapping f to g as in

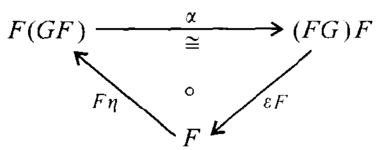
$$\frac{F_s X \xrightarrow{f} A}{X \xrightarrow[g]{} G_v A} \tag{1}$$

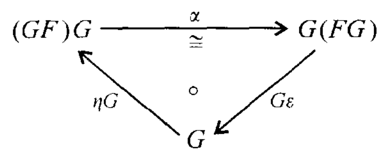
between $\boldsymbol{A}(F_s X, A)$ and $\boldsymbol{X}(X, G_v A)$, which is natural in X and A in the expected sense: for any $Y \in \boldsymbol{X}$, $t \in |F| Y$ and $h: X \to Y$ in addition to the above data, in

$$\frac{F_s X \xrightarrow{F_{st} h} F_t Y \xrightarrow{f} A}{X \xrightarrow[h]{} Y \longrightarrow G_v A}$$

we have $\varphi_{s,v}(f \circ F_{st} h) = \varphi_{t,v}(f) \circ h$, and similarly for data in $\boldsymbol{A}$.

We leave it to the reader to check that this is the same as the standard internal definition in the metabicategory ANACAT: the existence of $\eta : 1_X \to GF$ and $\varepsilon : FG \to 1_A$ such that



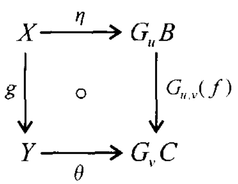


where the α's are the appropriate associativity isomorphisms. In particular, if $F \dashv G$, and $F' \cong F$, $G' \cong G$, then $F' \dashv G'$; and if $F \dashv G$, $F' \dashv G$, then $F' \cong F$.

Let $\boldsymbol{X} \overset{G}{\longleftarrow} \boldsymbol{A}$ be an anafunctor (in particular, G may be an ordinary functor), and $X \in \boldsymbol{X}$. We say that the triple $(B \in \boldsymbol{A}, u \in |G|\,B, \eta : X \to G_u B)$ is *good* for X if it has the universal property that for any $(A \in \boldsymbol{A}, v \in |G|\,A, g : X \to G_v A)$ there is a unique $f : B \to A$ with $g = G_{u,v}(f) \circ \eta$. $\boldsymbol{X} \overset{G}{\longleftarrow} \boldsymbol{A}$ satisfies the condition of *local existence of a left adjoint* if for every $X \in \boldsymbol{X}$, there is at least one good triple for X.

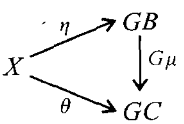
1. *Assume that the anafunctor $\boldsymbol{X} \overset{G}{\longleftarrow} \boldsymbol{A}$ satisfies the condition of local existence of a left adjoint. Then there is a (canonical) anafunctor $F : \boldsymbol{X} \to \boldsymbol{A}$ which is left adjoint to G.*

Proof. We define $F : \boldsymbol{X} \to \boldsymbol{A}$ as follows. For any $X \in \boldsymbol{X}$, $|F|\,X$ is the class of all good triples for X. If $s = (B, u, \eta) \in |F|\,X$, $F_s(X) \underset{\text{def}}{=} B$. If also $t = (C, v, \theta) \in |F|\,Y$, $g : X \to Y$, then $F_{s,t}(g)$ is the unique $f : B \to C$ such that



The bijection $\phi_{s,v}$ (see (1)) is as follows. If $s = (B, u, \eta) \in |F|\,X$ and $v \in |G|\,A$, for $f : F_s X \to A$, the corresponding $g : X \to G_v A$ is $g = G_{u,v}(f) \circ \eta$. The remaining details are similar to the ones in the basic theory of adjoint functors (see [11]).

When G is a functor, F constructed above is a saturated anafunctor. Indeed, given $s = (B, B, \eta) \in |F|\,X$ and $\mu : B \xrightarrow{\cong} C$, the condition for $t = (C, C, \theta) \in |F|\,X$ to satisfy $F_{s,t}(1_X) = \mu$ is that the diagram



commutes, which determines θ.

Let us also note that if $\boldsymbol{X}$, $\boldsymbol{A}$, G are all small, then so is F.

The example in 1.2 is, of course, a special case of 1, which is the main source of naturally occurring anafunctors.

Another special case of 1 says that any functor, or even anafunctor, which is fully faithful and essentially surjective has a quasi-inverse *anafunctor*; thus it is an equivalence (without the axiom of choice) in the sense of the metabicategory ANACAT. We call an anafunctor which is an equivalence in the sense of ANACAT an *anaequivalence* of categories. $F:\mathbf{X}\xrightarrow{a}\mathbf{A}$ is *fully faithful* if for every $X\in\mathbf{X}$ and $Y\in\mathbf{X}$, for some (equivalently, for all) $s\in|F|X$, $t\in|F|Y$, the mapping $F_{s,t}:\mathbf{X}(X,Y)\to\mathbf{A}(F_sX,F_tY)$ is a bijection. The same F is *essentially surjective* if for all $A\in\mathbf{A}$, there is $X\in\mathbf{X}$ and $s\in|F|X$ such that $A\cong F_sX$. We have

2. *Any fully faithful and essentially surjective (ana)functor is an anaequivalence of categories.*

By 1.11″ we have the following:

2′. *The inclusion* $\mathbf{A}\to\mathbf{A}^+$ $(=\mathrm{Ana}\,(\mathbf{1},\mathbf{A}))$ *is an anaequivalence.*

Completeness properties of functor-categories depend, in the usual treatment, on non-canonical choices. Assume $\mathbf{I}$, $\mathbf{X}$ and $\mathbf{A}$ are categories, and $\mathbf{A}$ has $\mathbf{I}$-indexed limits. Then the proof that the functor category $\mathrm{Fun}(\mathbf{X},\mathbf{A})$ has $\mathbf{I}$-indexed limits proceeds by picking particular limits in $\mathbf{A}$ of the $\mathbf{I}$-indexed diagrams in $\mathbf{A}$ obtained by evaluating the given $\mathbf{I}$-indexed diagram in $\mathrm{Fun}(\mathbf{X},\mathbf{A})$.

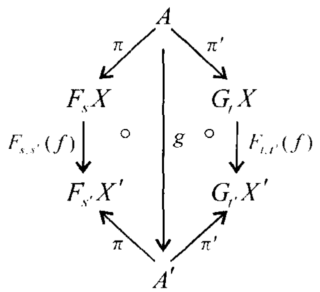
For the case when the category $\mathbf{I}$ has finitely many objects, we can avoid the choices. In fact, in this case the metacategory $\mathrm{ANA}(\mathbf{X},\mathbf{A})$ of anafunctors is *better* than the base category $\mathbf{A}$; it has *specified* limits (given as a function with arguments the $\mathbf{I}$-diagrams in $\mathbf{A}$) even if $\mathbf{A}$ is not assumed to have specified limits. We will have results concerning arbitrary small limit types $\mathbf{I}$; see propositions 6 and 7 below, and also the last section of the paper.

3. *Suppose that the small category* $\mathbf{I}$ *has finitely many objects, and the category* $\mathbf{A}$ *has* $\mathbf{I}$*-indexed limits. Then* $\mathrm{ANA}(\mathbf{X},\mathbf{A})$ *has specified* $\mathbf{I}$*-indexed limits.*

Proof. For simplicity of notation, we show why $\mathrm{ANA}(\mathbf{X},\mathbf{A})$ has specified binary products if $\mathbf{A}$ has binary products; the general case is only notationally different (but also see 4 below). Given F, $G\in\mathrm{ANA}(\mathbf{X},\mathbf{A})$, we define $F\times G\in\mathrm{ANA}(\mathbf{X},\mathbf{A})$ as follows. We put

$$|F\times G|X=\{(s\in|F|X,t\in|G|X,F_sX\xleftarrow{\pi}A\xrightarrow{\pi'}G_tX):(\pi,\pi')\text{ is a product in }\mathbf{A}\}.$$

For $a\in|F\times G|X$ as displayed, $(F\times G)_aX=A$. If also $a'\in|F\times G|X'$ with similar ingredients, and $f:X\to X'$, $(F\times G)_{a,a'}(f)$ is the arrow g in the following commutative diagram:



I leave it to the reader to define the projections $F \xleftarrow{\pi} F \times G \xrightarrow{\pi'} G$, and to check the universal property of the product.

We have the following variant of 3.

4. *Suppose that* $\boldsymbol{X}$, $\boldsymbol{A}$ *and* $\boldsymbol{I}$ *are small categories, and* $\boldsymbol{I}$ *has finitely many objects. Assume that* $\boldsymbol{A}$ *has* $\boldsymbol{I}$*-indexed limits. Then* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *has specified* $\boldsymbol{I}$*-indexed limits.*

Proof. By 1, we have $\mathrm{Lim}:\boldsymbol{A}^{\boldsymbol{I}}$ $(=\mathrm{Fun}(\boldsymbol{I},\boldsymbol{A})) \to \boldsymbol{A}$, an anafunctor right adjoint to $\Delta:\boldsymbol{A} \to \boldsymbol{A}^{\boldsymbol{I}}$. Since $\boldsymbol{A}$ is small, Lim is (can be taken to be) small; thus, the adjunction $\Delta \dashv \mathrm{Lim}$ lives in the bicategory AnaCat. As any bicategory, AnaCat has a representable functor to ANACAT, represented by any object of it:

$$\mathrm{Ana}(\boldsymbol{X},\ -) = \mathrm{AnaCat}(\boldsymbol{X},\ -):\mathrm{AnaCat} \to \mathrm{ANACAT}$$

(as explained before, we mean a homomorphism of bicategories when we talk about a functor of bicategories). As any functor of bicategories, $\mathrm{Ana}(\boldsymbol{X},\ -)$ preserves any adjunction in its domain. Thus, we have the adjunction

$$\mathrm{Ana}(\boldsymbol{X},\ \mathrm{Fun}(\boldsymbol{I},\boldsymbol{A})) \underset{\Delta^*}{\overset{\mathrm{Lim}^*}{\underset{\longleftarrow}{\overset{\longrightarrow}{\scriptstyle\top}}}} \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}), \tag{2}$$

where Lim^*, Δ^* are the *functors* $\mathrm{Ana}(\boldsymbol{X},\ \mathrm{Lim})$, $\mathrm{Ana}(\boldsymbol{X},\Delta)$, resp. We have the equivalences

$$\begin{aligned}
\mathrm{Ana}(\boldsymbol{X},\ \mathrm{Fun}(\boldsymbol{I},\boldsymbol{A})) &\underset{17}{\simeq} \mathrm{Ana}(\boldsymbol{X},\ \mathrm{Ana}(\boldsymbol{I},\boldsymbol{A})) \\
&\underset{14}{\simeq} \mathrm{Ana}(\boldsymbol{I},\ \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})) \underset{15}{\simeq} \mathrm{Fun}(\boldsymbol{I},\ \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})).
\end{aligned}$$

Composing them with (2), we get

$$\mathrm{Fun}(\boldsymbol{I}, \mathrm{Ana}(\boldsymbol{X}, \boldsymbol{A})) \underset{\hat{\Delta}}{\overset{\widehat{\mathrm{Lim}}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\boldsymbol{T}}}}} \mathrm{Ana}(\boldsymbol{X}, \boldsymbol{A}).$$

Going through the above equivalences, one can check that $\hat{\Delta}$ is isomorphic to $\Delta : \boldsymbol{B} \to \boldsymbol{B}^{\boldsymbol{I}}$ for $\boldsymbol{B} = \mathrm{Ana}(\boldsymbol{X}, \boldsymbol{A})$. Thus, up to isomorphism, $\mathrm{Lim}^{\wedge}$ is the desired limit-functor.

The conclusion of 4 holds, in particular, for $\boldsymbol{A}^{+} = \mathrm{Ana}(\mathbf{1}, \boldsymbol{A})$.

Of course, the similar result for colimits is a consequence, by passing to the opposite category. But also for other finitary categorical operations defined by universal properties, we have similar conclusions, at least for $\boldsymbol{A}^{+}$. E.g.,

5. *Suppose that the small category $\boldsymbol{A}$ is Cartesian closed. Then $\boldsymbol{A}^{+}$, the category of small anaobjects of $\boldsymbol{A}$* (a category anaequivalent to $\boldsymbol{A}$; see 2′), *is also Cartesian closed, and in fact has specified finite products and exponentials.*

Proof. An *exponential diagram on* a pair (X, Y) of objects in $\boldsymbol{A}$ is a diagram of the form

$$\begin{array}{ccccc} & Z & \xrightarrow{\quad e \quad} & Y \\ {}^{p}\swarrow & & \searrow^{q} & \\ X & & W & \end{array} \tag{3}$$

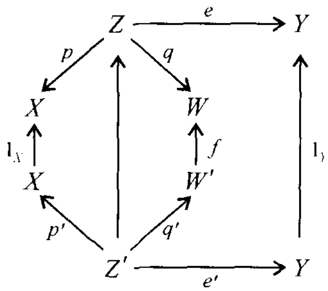
such that (p, q) is a product, and e satisfies the usual universal property of the evaluation morphism of an exponential (think of

$$\begin{array}{ccccc} & X \times Y^{X} & \xrightarrow{\quad e \quad} & Y \\ {}^{\pi}\swarrow & & \searrow^{\pi'} & \\ X & & Y^{X} & \end{array}$$

the definition is that for any

$$\begin{array}{ccccc} & Z' & \xrightarrow{\quad e' \quad} & Y \\ {}^{p'}\swarrow & & \searrow^{q'} & \\ X & & W' & \end{array}$$

such that (p', q') is a product, there is a unique commutative diagram of the form

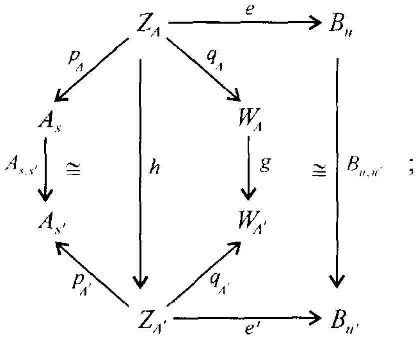


(4)

Of course, a category with finite products is Cartesian closed iff there exists an exponential diagram on any pair of objects.)

If Δ abbreviates (3), we indicate the components of Δ by putting the subscript Δ to the corresponding symbol in (3); e.g., W_Δ for the object W in (3), etc.

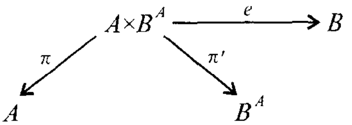
Let A, B be anaobjects of $\mathbb{A}$. Define the anaobject B^A as follows. Let $|B^A|$ be the set of all (s, u, Δ) such that $s \in |A|$, $u \in |B|$, and Δ is an exponential diagram on (A_s, B_u). For $a = (s, u, \Delta) \in |B^A|$, let $(B^A)_a \underset{\text{def}}{=} W_\Delta$. Here and below, $a = (s, u, \Delta) \in |B^A|$ and $a' = (s', u', \Delta') \in |B^A|$. $(B^A)_{a,a'} : W_\Delta \to W_{\Delta'}$ is defined to be the arrow g in the unique commutative diagram



; (5)

the reasons why the latter uniquely exists are the universal property of Δ', and the fact that $A_{s,s'}$, $B_{u,u'}$ are isomorphisms.

The exponential diagram



on (A, B) is given as follows. $|A \times B^A| \underset{\text{def}}{=} |B^A|$; $(A \times B^A)_a = Z_\Delta$; $(A \times B^A)_{a,a'}$ is the arrow h in (5). For $t \in |A|$, $\pi_{a,t} : Z \to A_{s'}$ is $A_{s,t} \circ p_\Delta$; π' is similar. For $v \in |B|$, $e_{a,v} : Z \to B_v$ is $B_{u,v} \circ e_\Delta$.

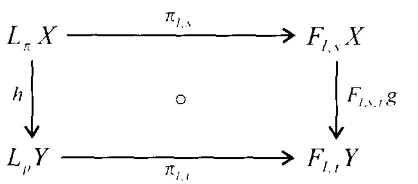
The verification of the needed properties of these data is omitted.

6. *Let* $\boldsymbol{X}$ *be a small category,* $\boldsymbol{A}$ *a category having all small limits. The every small diagram in* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *has a limit in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$; that is, with $\varphi:\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *the inclusion, for any small* $\boldsymbol{I}$ *and* $\Gamma:\boldsymbol{I} \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, $\lim(\varphi \circ \Gamma)$ *exists in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$. *Moreover, there is a class-function assigning, to any small diagram* Γ *in* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$, *a limit-cone in* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *on* $\varphi \circ \Gamma$. *If* $\boldsymbol{A}$ *is locally small, the limit-objects in the assigned limit-cones are locally small anafunctors.*

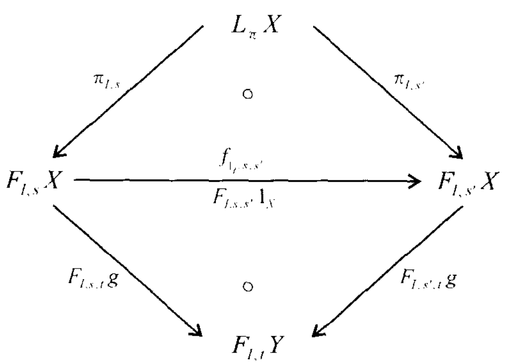
Proof. Let $\Gamma = (\langle F_I \rangle_{I\in I}, \langle f_i : F_I \to F_J \rangle_{(i:I\to J)\in I})$ be a small diagram in $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$. We define $L = \lim \Gamma \in \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ as follows.

Fix $X \in \boldsymbol{X}$, to define $|L|\,X$. We let $\boldsymbol{I}|X$ be the category whose objects are pairs (I, s) with $I \in \boldsymbol{I}$ and $s \in |F_I|\,X$, and whose arrows $(I,s) \to (J,t)$ are (s,t,i) with $i:I \to J$ (that is, an arrow $(I,s) \to (J,t)$ is just an arrow $I \to J$, with the information on the domain (I,s) and the codomain (J,t) attached; we will write $i:(I,s) \to (J,t)$ instead of (s,t,i): $(I,s) \to (J,t)$). By the hypotheses, $\boldsymbol{I}|X$ is a small category. Consider the diagram $\Gamma|X:\boldsymbol{I}|X \to \boldsymbol{A}$ that assigns the object $F_{I,s}X \equiv (F_I)_s(X)$ to (I,s), and the arrow $f_{i,s,t} \equiv (f_i)_{s,t}:F_{I,s}X \to F_{J,t}X$ to $i:(I,s) \to (J,t)$. We define $|L|\,X$ to be the class of all limit-cones on $\Gamma|X$ in $\boldsymbol{A}$; for $\pi \in |L|\,X$, $\pi = \langle \pi_{I,s}:[\pi] \to F_{I,s}X\rangle_{(I,s)\in \Gamma|X}$, we put $L_\pi(X) = [\pi]$.

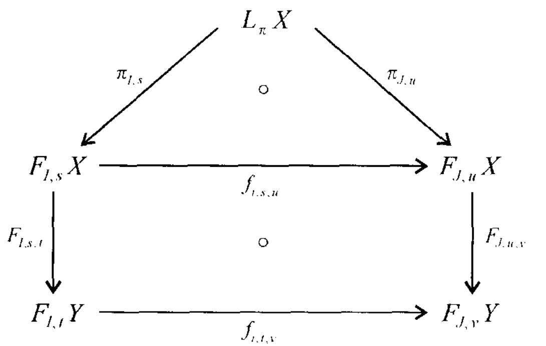
Let $g:X \to Y$ be an arrow, $\pi \in |L|\,X$, $\rho \in |L|\,Y$, to define $h \equiv L_{\pi,\rho}(g)$: $L_\pi(X) \to L_\rho(X)$. h is given uniquely by the condition that



commutes for all $I \in \boldsymbol{I}$, $s \in |F|\,X$, $t \in |F|\,Y$. Indeed, first of all, the diagram



shows that the arrow $k_{I,t} \underset{\text{def}}{=} F_{I,s,t}g \circ \pi_{I,s}:L_\pi X \to F_{I,t}Y$ does not depend on s (the upper commutativity is by π being a cone, the lower by the functoriality of F_I; the equality $f_{1_{I},s,s'} = (1_{F_I})_{s,s'} = F_{s,s'}(1_X)$ holds by the compatibility of the diagram Γ, and the definition of 1_{F_I}). Next, the diagram



shows that $\langle k_{I,t}\rangle_{(I,t)\in \mathrm{Ob}(I|Y)}$ is cone on the diagram $\Gamma|Y$. Since $\langle \rho_{I,t}: L_\rho Y \to F_{I,t} Y\rangle_{I,t}$ is a limit cone, there is a unique $h: L_\pi X \to L_\rho Y$ such that $h \circ \rho_{I,t} = k_{I,t}$ for all I and t, which is our assertion on h.

Having defined $L_{\pi,\rho}(g)$, I leave it to the reader to check that L so defined is indeed an anafunctor. We have $\lambda_I: L \to F_I$ for which $\lambda_{I,\pi,s} = \pi_{I,s}$, for all appropriate values of the parameters; moreover, $\langle \lambda_I \rangle_I$ is a limit cone on the diagram $\varphi \circ \Gamma$; the verification is omitted.

Note that, in the proof, in order to build the required $\boldsymbol{I}$-type limit, we use a whole class of other limit-types, to construct limits in A. However, when each F_I is in particular a functor, than each $\boldsymbol{I}|X$ is isomorphic to $\boldsymbol{I}$; this shows that we have

7. *Assuming that $\boldsymbol{A}$ has $\boldsymbol{I}$-type limits, then $\boldsymbol{I}$-type diagrams of functors $\boldsymbol{X} \to \boldsymbol{A}$ have specified limits in* ANA$(\boldsymbol{X},\boldsymbol{A})$.

The last observation is due to the referee.

3. Anabicategories

In a two-dimensional category, for a given pair of objects (0-cells), the totality of arrows (1-cells) from one to the other form a category; in short, arrows are objects in categories, and thus, *one should attempt to determine them only up to isomorphism.* This means that, for a given triple of 0-cells A, B, C composition of arrows $A \to B$ with arrows $B \to C$, instead of being a functor as it is in a bicategory, should be an anafunctor, preferably a saturated one.

The definition of *anabicategory* is obtained from that of bicategory by natural modifications, amounting to replacing functors by anafunctors in all places. There is a good abstract way of saying this: an anabicategory is ANACAT-*enriched bicategory.* I will not attempt to make this completely explicit, although I believe this is the right way of looking at the concept. From the description that follows, the reader may get a reasonably good idea of the notion of an *$\mathscr{V}$-enriched bicategory*, for any Cartesian (in fact, for any *monoidal*) bicategory $\mathscr{V}$. The point is that *the more general notion* ($\mathscr{V}$-enriched bicategory) *is easier to grasp than the more concrete one* (ANACAT-enriched bicategory), because there are fewer elements of the abstract situation available for manipulation than there are in the concrete one. This was already fondly pointed out by me in [12] in the context of the notion of 2-category.

An *anabicategory* $\mathscr{A}$ has data 1 (i)–(vi), satisfying conditions (vii) and (viii):

1. (i) *Objects* (0-cells).

(ii) For any pair of objects A, B a category $\mathscr{A}(A, B)$, or simply $[A, B]$, of *arrows* (1-cells) $A \to B$ (the objects of $[A, B]$), and *2-cells* as α in $A \underset{g}{\overset{f}{\rightrightarrows}} B$ ($\downarrow\alpha$) (the arrows of the category $[A, B]$).

(iii) For any object A, an *identity* anaobject of $[A, A]$ (an anafunctor $1_A : \mathbf{1} \to [A, A]$).

(iv) For any objects A, B, C, an anafunctor

$$\circ_{A,B,C} : [A, B] \times [B, C] \to [A, C]$$

(*composition*).

(v) $\lambda_{A,B} : 1_B \circ (\ ) \xrightarrow{\cong} 1_{[A,B]}$ and $\rho_{A,B} : (\ ) \circ 1_A \xrightarrow{\cong} 1_{[A,B]}$ (*left* and *right identity isomorphisms*); here, $1_B \circ (\ )$ is an abbreviated notation for the composite (in ANACAT) of

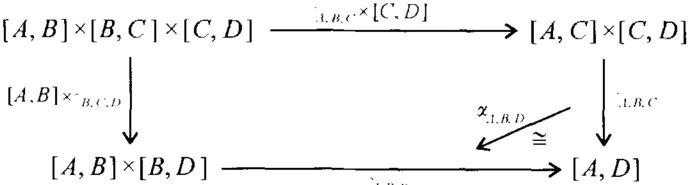
$$[A, B] \xrightarrow{\cong} [A, B] \times \mathbf{1} \xrightarrow{[A,B] \times 1_B} [A, B] \times [B, B] \xrightarrow{\circ_{A,B,B}} [A, B];$$

the natural transformations $\lambda = \lambda_{A,B}$, $\rho = \rho_{A,B}$ have components

$$\lambda_{a;j} : 1_{B,j} \circ_a f \xrightarrow{\cong} f, \qquad \rho_{b;i} : f \circ_b 1_{A,i} \xrightarrow{\cong} f$$

$(f : A \to B,\ i \in |1_A|,\ j \in |1_B|,\ a \in |\circ_{A,B,B}|(f, 1_{B,j}),\ b \in |\circ_{A,A,B}|(1_{A,i}, f))$.

(vi) For any objects A, B, C, D the *associativity isomorphism*, a 2-cell $\alpha_{A,B,C,D}$ between composite arrows as shown in



The natural transformation $\alpha = \alpha_{A,B,C,D}$ has components

$$\alpha_{a,b,c,d} : h \circ_b (g \circ_a f) \xrightarrow{\cong} (h \circ_c g) \circ_d f$$

$(f : A \to B,\quad g : B \to C,\quad h : C \to D,\quad a \in |\circ_{A,B,C}|(f, g),\quad b \in |\circ_{A,C,D}|(g \circ_a f, h),$
$c \in |\circ_{B,C,D}|(g, h),\ d \in |\circ_{A,B,D}|(f, h \circ_c g))$.

Remark. To simplify notation, here and below we ignore the canonical equivalences and isomorphisms that are part of the Cartesian bicategorical structure of ANACAT. Thus, when we want to state the notation of "$\mathcal{V}$-enriched bicategory", there is another object in the diagram, and the left-hand side looks like

$$\begin{array}{c} ([A,B]\otimes[B,C])\otimes[C,D] \\ \simeq\big\downarrow\hat{p} \\ [A,B]\otimes([B,C]\otimes[C,D]) \\ {\scriptstyle [A,B]\otimes\circ_{B,C,D}}\big\downarrow \\ [A,B]\otimes[B,D] \end{array}$$

with $\hat{p}$ the appropriate canonical equivalence in $\mathcal{A}$. In ANACAT, $\hat{p}$ is in fact an isomorphism.

(vii) (*Associativity coherence*). Let A, B, C, D and E be objects. With a logic that should be easy to guess, we make the following abbreviations:

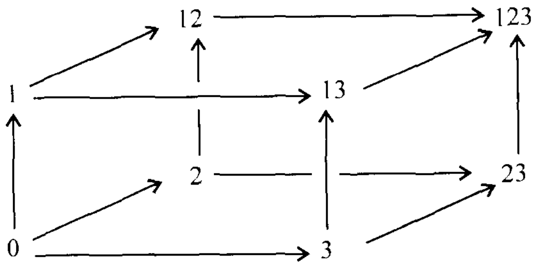
$$\begin{aligned} 0 &= [A,B]\times[B,C]\times[C,D]\times[D,E] \\ 1 &= [A,C]\times[C,D]\times[D,E] \\ 2 &= [A,B]\times[B,D]\times[D,E] \\ 3 &= [A,B]\times[B,C]\times[C,E] \\ 12 &= [A,D]\times[D,E] \\ 13 &= [A,C]\times[C,E] \\ 23 &= [A,B]\times[B,E] \\ 123 &= [A,E] \end{aligned}$$

$$\begin{aligned} \langle 0,1\rangle &= \circ_{A,B,C}\times[C,D]\times[D,E] &&: 0\to 1 \\ \langle 0,2\rangle &= [A,B]\times\circ_{B,C,D}\times[D,E] &&: 0\to 2 \\ \langle 0,3\rangle &= [A,B]\times[B,C]\times\circ_{C,D,E} &&: 0\to 3 \\ \langle 1,12\rangle &= \circ_{A,C,D}\times[D,E] &&: 1\to 12 \\ \langle 1,13\rangle &= [A,C]\circ_{C,D,E} &&: 1\to 13 \\ \langle 2,12\rangle &= \circ_{A,B,D}\times[D,E] &&: 2\to 12 \\ \langle 2,23\rangle &= [A,B]\times\circ_{B,D,E} &&: 2\to 23 \\ \langle 3,13\rangle &= \circ_{A,B,C}\times[C,E] &&: 3\to 13 \\ \langle 3,23\rangle &= [A,B]\times\circ_{B,C,E} &&: 3\to 23 \\ \langle 12,123\rangle &= \circ_{A,D,E} &&: 12\to 123 \end{aligned}$$

$\langle 13, 123\rangle \ = \circ_{A,C,E} : 13 \to 123$

$\langle 23, 123\rangle \ = \circ_{A,B,E} : 23 \to 123$

We have the cube



with the edges the anafunctors indicated. Each face of the cube has a 2-cell in it as follows:

The front face:

$$(13) = \mathrm{id} : \langle 1, 13\rangle\langle 0, 1\rangle \to \langle 3, 13\rangle\langle 0, 3\rangle$$

(both composites are canonically isomorphic to $\circ_{A,B,C} \times \circ_{C,D,E}$; taking the products in ANACAT as in CAT, the isomorphism is the identity);

The back face:

$$(13)2 = \alpha_{A,B,D,E} : \langle 12, 123\rangle\langle 2, 12\rangle \to \langle 23, 123\rangle\langle 2, 23\rangle;$$

The bottom face:

$$(23) = [A, B] \times \alpha_{B,C,D,E} : \langle 2, 23\rangle\langle 0, 2\rangle \to \langle 3, 23\rangle\langle 0, 3\rangle;$$

The top face:

$$(23)1 = \alpha_{A,C,D,E} : \langle 12, 123\rangle\langle 1, 12\rangle \to \langle 13, 123\rangle\langle 1, 13\rangle;$$

The left face:

$$(12) = \alpha_{A,B,C,D} \times [D, E] : \langle 1, 12\rangle\langle 0, 1\rangle \to \langle 2, 12\rangle\langle 0, 2\rangle;$$

The right face:

$$(12)3 = \alpha_{A,B,C,E} : \langle 13, 123\rangle\langle 3, 13\rangle \to \langle 23, 123\rangle\langle 3, 23\rangle.$$

With triple composites meant as associated to the left, there are six composite anafunctors from 0 to 123:

$$\langle 123\rangle = \langle 12, 123\rangle\langle 1, 12\rangle\langle 0, 1\rangle,$$

$$\langle 132\rangle = \langle 13, 123\rangle\langle 1, 13\rangle\langle 0, 1\rangle,$$

$$\langle 213\rangle = \langle 12, 123\rangle\langle 2, 12\rangle\langle 0, 2\rangle,$$

$$\langle 231\rangle = \langle 23, 123\rangle\langle 2, 23\rangle\langle 0, 2\rangle,$$

$$\langle 312\rangle = \langle 13, 123\rangle\langle 3, 13\rangle\langle 0, 3\rangle,$$

$$\langle 321\rangle = \langle 23, 123\rangle\langle 3, 23\rangle\langle 0, 3\rangle;$$

and six composite natural transformations between the latter, which (after ignoring canonical isomorphisms in ANACAT) are as follows:

$$\bar{1}23 = (12) \circ \langle 12, 123 \rangle \;: \langle 123 \rangle \to \langle 213 \rangle,$$

$$1\bar{2}3 = \langle 0, 1 \rangle \circ (23)1 \;: \langle 123 \rangle \to \langle 132 \rangle,$$

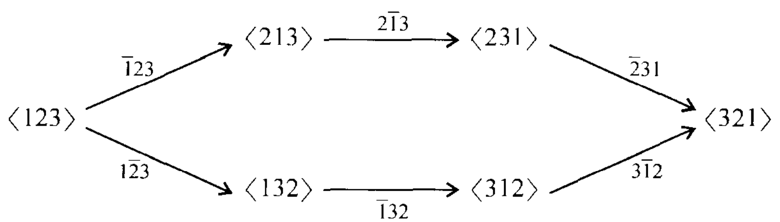
$$2\bar{1}3 = \langle 0, 2 \rangle \circ (13)2 \;: \langle 213 \rangle \to \langle 231 \rangle,$$

$$\bar{2}31 = \langle 23, 123 \rangle \circ (23) \;: \langle 231 \rangle \to \langle 321 \rangle,$$

$$\bar{1}32 = \langle 13, 123 \rangle \circ (13) \;: \langle 132 \rangle \to \langle 312 \rangle,$$

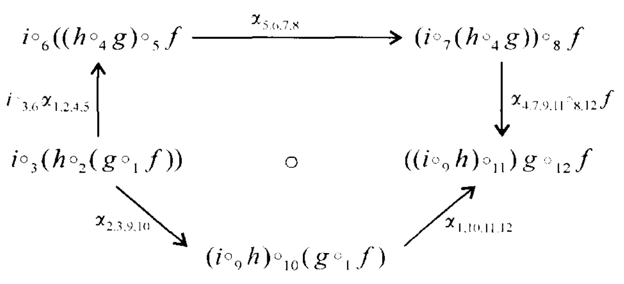
$$3\bar{1}2 = \langle 0, 3 \rangle \circ (12)3 \;: \langle 312 \rangle \to \langle 321 \rangle.$$

We obtain the following diagram of natural transformations:



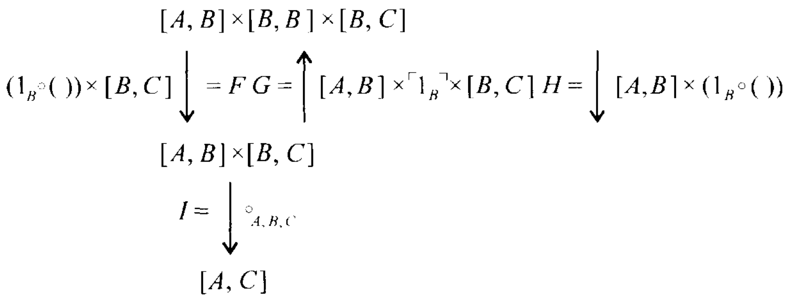
The associativity–coherence condition is that the last diagram be commutative, for any choice of A, B, C, D and E. This is essentially a pentagon since $\bar{1}32$ is (can be taken to be) the identity; the above way of presenting it is more symmetric.

When we present the last diagram componentwise, we take a 4-tuple ($f: A \to B$, $g: B \to C$, $h: C \to D$, $i: D \to E$) of arrows, and compute their (five) composites with all possible bracketings. Since there are twelve distinct compositions to be performed (corresponding to the twelve edges of the cube), we will use twelve arbitrary specifications for ten composition anafunctors (ten because any one of them is determined by selecting three out of the five given objects); two of them are used twice (see the formulas above for the composites $0 \to 123$).The condition becomes the coherence pentagon ((1.1) (A.C.) in [1], p.6), with the twelve arbitrary specifications appropriately added; explicitly

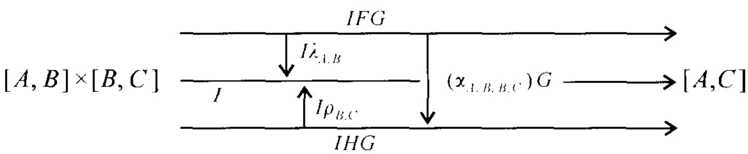


with the numbers 1 to 12 standing for appropriate, but otherwise arbitrary, specifications.

(viii) (*Identity coherence*) Consider



and



The condition is the commutativity

$$\begin{array}{ccc} IFG & \xrightarrow{(\alpha_{A,B,B,C})G} & IHG \\ & {\scriptstyle I\lambda_{A,B}}\searrow \quad \circ \quad \swarrow{\scriptstyle I\rho_{B,C}} & \\ & I & \end{array}$$

In components, this becomes

$$\begin{array}{ccc} g\circ_b(1_{B,i}\circ_a f) & \xrightarrow{\alpha_{a,b,c,d}} & (g\circ_c 1_{B,i})\circ_d f) \\ & {\scriptstyle g\circ_{b,e}\lambda_{a;i}}\searrow \quad \circ \quad \swarrow{\scriptstyle \rho_{c;i}\circ_{d,e}f} & \\ & g\circ_e f & \end{array}$$

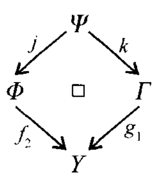
($f: A \to B$, $g: B \to C$, etc.).

A one-object bicategory is a monoidal category; its objects are the 1-cells of the bicategory, arrows the 2-cells, tensor-product is composition of 1-cells, the (two-sided) unit I is the identity 1-cell 1_*. A one-object anabicategory is an *anamonoidal category*.

An anamonoidal category is a category $\mathbf{A}$, with, among others, a unit anaobject $I = \mathrm{I}_{\mathbf{A}}$ of $\mathbf{A}$ (that is, an anafunctor $I:\mathbf{1} \to \mathbf{A}$) and a tensor-product anafunctor $\otimes\, \mathbf{A}\times\mathbf{A} \to \mathbf{A}$; for objects A, B, we have tensors $A\otimes_a B$, for any specification $a \in |\otimes|(A,B)$. (Note the changed order in $A\otimes_a B$ of the arguments with respect to the notation of composites.) To say that $\otimes$ is saturated is to say that whenever $\mu: A\otimes_a B \xrightarrow{\cong} C$, there is a unique $b \in |\otimes|(A,B)$ such that $C = A\otimes_b B$ and $\mu: 1_A \otimes_{a,b} 1_B$; if so, and if I is saturated too, we say that $\mathbf{A}$ is a *saturated* anamonoidal category.

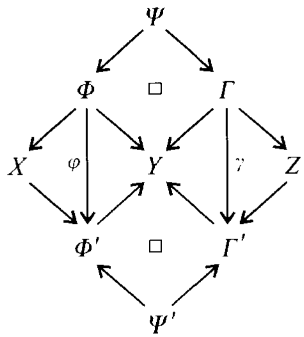
2. Example. Let R be a commutative ring (with 1); Mod_R the category of R-modules. Mod_R is a saturated anamonoidal category, with the following canonical structure. $|\otimes|(A,B)$ is the set of all entities $a = (C, \eta:(A,B) \to C)$ where η is a universal (R-)bilinear map from (A,B): for any bilinear $\varphi:(A,B) \to D$, there is unique $f: C \to D$ such that $\varphi = f\circ\eta$; we put $A\otimes_a B = C$. Given also $b = (F, \theta:(D,E) \to F) \in |\otimes|(D,E)$, and $f: A \to D$, $g: B \to E$, $f\otimes_{a,b}\ g: A\otimes_a B \to D\otimes_b E$ is the unqiue arrow $h: C \to F$ for which $\theta = f\circ\eta$. The unit anaobject is the saturation $\ulcorner R\urcorner^{\#}:\mathbf{1} \to \mathrm{Mod}_R$ of $\ulcorner R\urcorner:\mathbf{1} \to \mathrm{Mod}_R$ picking out R as a module over itself. I leave the definition of the identity and associativity isomorphisms to the reader. This is the usual definition of Mod_R as a monoidal category, except for the absence of a non-canonical choice of the tensors, and a somewhat artificial-looking way of dealing with the unit object. (We could leave the unit object as it usually is, given by $\ulcorner R\urcorner:\mathbf{1} \to \mathrm{Mod}_R$ (as an anafunctor), but then Mod_R is not saturated; we will see the advantages of saturation for anamonoidal categories when we discuss morphisms between them.) The usual verification of the monoidal properties works with no essential change to show that we indeed have an anamonoidal category. It is in fact saturated; the part of this fact that concerns the tensor-operation is essentially equivalent to saying that the universal bilinear map is unique up to a unique isomorphism.

3. Example. Given a category $\mathbf{X}$ with (not necessarily specified) pullbacks, we have the anabicategory $\mathrm{Span}(\mathbf{X})$ of *spans* in $\mathbf{X}$, defined as follows. The objects are the objects of $\mathbf{X}$, an arrow $f: X \to Y$ in $\mathrm{Span}(\mathbf{X})$ is a pair $f = (X \xleftarrow{f_1} \Phi \xrightarrow{f_2} Y)$ of arrows in $\mathbf{X}$; with also $f' = (X \xleftarrow{f_1'} \Phi' \xrightarrow{f_2'} Y): X \to Y$, a 2-cell $\varphi: f \to f'$ is an arrow in $\varphi: \Phi \to \Phi'$ with $f_1'\varphi = f_1$, $f_2' j = f_2$; composition of 2-cells is composition in $\mathbf{X}$. With f as above, and $g = (Y \xleftarrow{g_1} \Gamma \xrightarrow{g_2} Z): Y \to Z$, we define $|\circ_{X,Y,Z}|\,(f,g)$ as the set of pullback diagrams

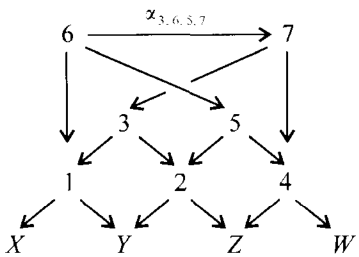


; (1)

denoting (1) by a, we define $g \circ_a f$ to be $(X \xleftarrow{f_1 j} \Psi \xrightarrow{g_2 k} Z): X \to Z$. Given, in addition to data as above, also $g' = (Y \xleftarrow{g'_1} \Gamma' \xrightarrow{g'_2} Z) : Y \to Z$, $a' \in |\circ_{X,Y,Z}|(f', g')$, $\gamma : g \to g'$, $\varphi \circ_{a,a'} \gamma : g \circ_a f \to g' \circ_{a'} f'$ is defined by the arrow $\psi : \Psi \to \Psi'$ we can put in the diagram



by the universal property of the lower pullback. The identity $1_X : X \to X$ is given by the span $(X \xleftarrow{1_X} X \xrightarrow{1_X} X)$ (the identity anaobject of Span $(\boldsymbol{X})$ (X, X) is now an ordinary object). The identity and associativity isomorphisms for the composition are given by canonical isomorphisms of pullbacks; the picture



(2)

contains the construction of the associativity isomorphism

$$\alpha_{3,6,5,7} : 4 \circ_6 (2 \circ_3 1) \xrightarrow{\cong} (4 \circ_5 2) \circ_7 1;$$

in (2), each of the items 3, 5, 6 and 7 is obtained by taking a pullback. The rest of the definition and the verification that all this gives an anabicategory are left to the reader. In fact, the definition is the usual one ((2.6) in [1]) for the bicategory of spans, with the non-canonical choices removed. The composition functors are saturated in this example.

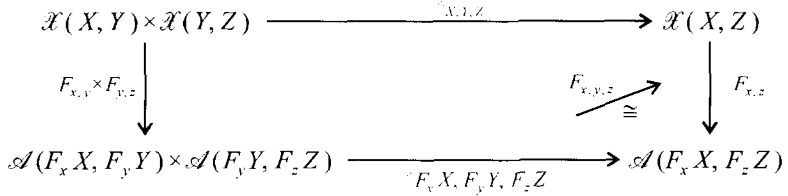
A morphism of anabicategories will be called an *anafunctor* (*of anabicategories*); this is the natural counterpart of "homomorphism of bicategories" (see [1]), or functor of bicategories as we called it above. With $\mathscr{X}$ and $\mathscr{A}$ anabicategories, an anafunctor $F : \mathscr{X} \to \mathscr{A}$ is given by data 4(i)–(iv), satisfying 4 (v) and (vi), all given below. Again, we actually have a more general notion, that of an anafunctor of $\mathscr{V}$-enriched bicategories.

4. (i) A class $|F|$, with maps $\sigma : |F| \to |\mathbf{X}|$ ("source"), $\tau : |F| \to |\mathbf{A}|$ ("target"). $|F|$ is the class of *specifications*; $x \in |F|$ "specifies the value $\tau(x)$ at the argument $\sigma(x)$". For $X \in \mathbf{X}$, we write $|F|\,X$ for the class $\{x \in |F| : \sigma(x) = X\}$, and $F_x(X)$ for $\tau(x)$; the notation $F_x(X)$ presumes that $x \in |F|\,X$. As before, we write $|F|(X, A)$ for $\{x \in |F|\,X : F_x(X) = A\}$.

I note that it would be natural to make $|F|\,(X, A)$ into a category; indeed, *saturation* for F will mean an *additional structure*, with $|F|\,(X, A)$ made the object-class of a groupoid; I will not pursue this in this paper, but see [13]).

(ii) For any $x \in |F|\,X$, $y \in |F|\,Y$, an anafunctor $F_{x,y} : \mathscr{X}(X, Y) \to \mathscr{A}(F_x X, F_y Y)$.

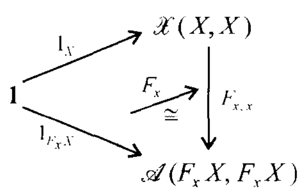
(iii) (*Composition isomorphism*) For x and y as before, and $z \in |F|\,Z$, an isomorphism natural transformation $F_{x,y,z}$ as in



The components of $F_{x,y,z}$ are 2-cells of the form

$$F_{x,y,z;s,t,u;a,b} : (F_{y,z;t}\, g) \circ_b (F_{x,y;s}\, f) \xrightarrow{\cong} F_{x,z;u}(g \circ_a f). \tag{3}$$

(iv) (*Identity isomorphism*) For any $X \in \mathscr{X}$ and $x \in |F|\,X$, an isomorphism F_x as in



the components of F_x are of the form $F_{x;u,i,j} : 1_{F_x X, i} \xrightarrow{\cong} F_{x,x;u}(1_{X,i})$.

(v) (*Composition coherence*) We arrive at the coherence condition for the transition isomorphisms $F_{x,y,z}$ without explicit use of further specifications by drawing diagrams of anafunctors and natural transformations in the style of 1 (vii). Assume

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

and

$$x \in |F|\,X,\ y \in |F|\,Y,\ z \in |F|\,Z,\ w \in |F|\,W.$$

Let us abbreviate:

$$\bar{X} = F_x X, \qquad \bar{Y} = F_y Y, \qquad \bar{Z} = F_z Z, \qquad \bar{W} = F_w W,$$

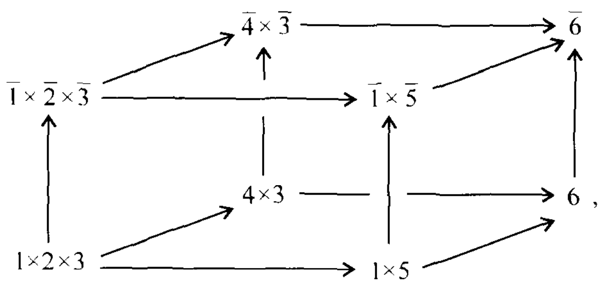
$$1 = \mathscr{X}(X, Y), \qquad 2 = \mathscr{X}(Y, Z), \qquad 3 = \mathscr{X}(Z, W),$$

$$4 = \mathscr{X}(X, Z), \qquad 5 = \mathscr{X}(Y, W), \qquad 6 = \mathscr{X}(X, W),$$

$$\bar{1} = \mathscr{A}(\bar{X}, \bar{Y}), \qquad \bar{2} = \mathscr{A}(\bar{Y}, \bar{Z}), \qquad \bar{3} = \mathscr{A}(\bar{Z}, \bar{W}),$$

$$\bar{4} = \mathscr{A}(\bar{X}, \bar{Z}), \qquad \bar{5} = \mathscr{A}(\bar{Y}, \bar{W}), \qquad \bar{6} = \mathscr{A}(\bar{X}, \bar{W})$$

and consider



with edges

$$\langle 1\rangle = F_{x,y} \times F_{y,z} \times F_{z,w} : 1 \times 2 \times 3 \to \bar{1} \times \bar{2} \times \bar{3},$$

$$\langle 2\rangle = \mathscr{X}(X, Y) \times \circ_{Y,Z,W} : 1 \times 2 \times 3 \to 1 \times 5,$$

$$\langle 3\rangle = \mathscr{A}(\bar{X}, \bar{Y}) \times \circ_{\bar{Y},\bar{Z},\bar{W}} : \bar{1} \times \bar{2} \times \bar{3} \to \bar{1} \times \bar{5},$$

$$\langle 4\rangle = F_{x,y} \times F_{y,w} : 1 \times 5 \to \bar{1} \times \bar{5},$$

$$\langle 5\rangle = \circ_{X,Y,Z} \times \mathscr{X}(Z, W) : 1 \times 2 \times 3 \to 4 \times 3,$$

$$\langle 6\rangle = \circ_{\bar{X},\bar{Y},\bar{Z}} \times \mathscr{A}(\bar{Z}, \bar{W}) : \bar{1} \times \bar{2} \times \bar{3} \to \bar{4} \times \bar{3},$$

$$\langle 7\rangle = F_{x,z} \times F_{z,w} : 4 \times 3 \to \bar{4} \times \bar{3},$$

$$\langle 8\rangle = \circ_{X,Z,W} : 4 \times 3 \to 6,$$

$$\langle 9\rangle = \circ_{X,Y,W} : 1 \times 5 \to 6,$$

$$\langle 10\rangle = \circ_{\bar{X},\bar{Y},\bar{W}} : \bar{1} \times \bar{5} \to \bar{6},$$

$$\langle 11\rangle = F_{X,W} : 6 \to \bar{6},$$

$$\langle 12\rangle = \circ_{\bar{X},\bar{Z},\bar{W}} : \bar{4} \times \bar{3} \to \bar{6}.$$

The front and left faces are themselves composites as follows:

$$\begin{array}{ccc} \bar{1}\times\bar{2}\times\bar{3} & \xrightarrow{\langle 3\rangle} & \bar{1}\times\bar{5} \\ \uparrow{\scriptstyle\langle 1.2\rangle} & & {\scriptstyle\langle 4.2\rangle}\uparrow \\ 1\times\bar{2}\times\bar{3} & \xrightarrow{\langle 13\rangle} & 1\times\bar{5} \\ \uparrow{\scriptstyle\langle 1.1\rangle} & & {\scriptstyle\langle 4.1\rangle}\uparrow \\ 1\times 2\times 3 & \xrightarrow[\langle 2\rangle]{} & 1\times 5 \end{array} \qquad \begin{array}{ccc} \bar{1}\times\bar{2}\times\bar{3} & \xrightarrow{\langle 6\rangle} & \bar{4}\times\bar{3} \\ \uparrow{\scriptstyle\langle 1.4\rangle} & & {\scriptstyle\langle 7.2\rangle}\uparrow \\ 1\times 2\times\bar{3} & \xrightarrow{\langle 14\rangle} & 4\times\bar{3} \\ \uparrow{\scriptstyle\langle 1.3\rangle} & & {\scriptstyle\langle 7.1\rangle}\uparrow \\ 1\times 2\times 3 & \xrightarrow[\langle 5\rangle]{} & 4\times 3 \end{array}$$

where the arrows are the ones obtained by the above pattern. The faces of the cube each have an isomorphism 2-cell in them as follows:

The lower front face:

$$\mathscr{X}(X,Y)\times F_{y,z,w} : \langle 13\rangle\langle 1.1\rangle \to \langle 4.1\rangle\langle 2\rangle;$$

The upper front face:

$$\mathrm{id} : F_{x,y}\times \circ_{\bar{Y},\bar{Z},\bar{W}} \to F_{x,y}\times \circ_{\bar{Y},\bar{Z},\bar{W}};$$

The (complete) front face:

$$[1] = (F_{x,y}\times \circ_{\bar{Y},\bar{Z},\bar{W}})(\mathscr{X}(X,Y)\times F_{y,z,w}) : \langle 3\rangle\langle 1\rangle \to \langle 4\rangle\langle 2\rangle;$$

The back face:

$$[2] = F_{x,z,w} : \langle 15\rangle\langle 10\rangle \to \langle 14\rangle\langle 11\rangle;$$

The bottom face:

$$[3] = \alpha_{X,Y,Z,W} : \langle 11\rangle\langle 8\rangle \to \langle 12\rangle\langle 2\rangle;$$

The top face:

$$[4] = \alpha_{\bar{X},\bar{Y},\bar{Z},\bar{W}} : \langle 15\rangle\langle 9\rangle \to \langle 13\rangle\langle 6\rangle;$$

The lower left face:

$$\mathrm{id} : \circ_{X,Y,Z}\times F_{z,w} \to \circ_{X,Y,Z}\times F_{z,w};$$

The upper left face:

$$F_{x,y,z}\times \mathscr{A}(\bar{Z},\bar{W}) : \langle 6\rangle\langle 1.4\rangle \to \langle 7.2\rangle\langle 14\rangle;$$

The (complete) left face:

$$[5] = (F_{x,y,z}\times \mathscr{A}(\bar{Z},\bar{W}))(\circ_{X,Y,Z}\times F_{z,w}) : \langle 6\rangle\langle 1\rangle \to \langle 7\rangle\langle 5\rangle;$$

The right face:

$$[6] = F_{x,y,w} : \langle 10\rangle\langle 4\rangle \to \langle 11\rangle\langle 9\rangle.$$

We have six composite arrows from $1 \times 2 \times 3$ to $\bar{6}$ in the cube, and six 2-cells between them; the commutativity of the resulting hexagon is the requirement:

$$\begin{array}{ccccccc}
 & & \langle 10\rangle\langle 3\rangle\langle 1\rangle & \xrightarrow{\langle 10\rangle[1]} & \langle 10\rangle\langle 4\rangle\langle 2\rangle & & \\
 & {\scriptstyle [4]\langle 1\rangle}\nearrow & & & & \searrow{\scriptstyle [6]\langle 2\rangle} & \\
\langle 12\rangle\langle 6\rangle\langle 1\rangle & & & \circ & & & \langle 11\rangle\langle 9\rangle\langle 2\rangle \\
 & {\scriptstyle \langle 12\rangle[5]}\searrow & & & & \nearrow{\scriptstyle \langle 11\rangle[3]} & \\
 & & \langle 12\rangle\langle 7\rangle\langle 5\rangle & \xrightarrow[{[2]\langle 5\rangle}]{} & \langle 11\rangle\langle 8\rangle\langle 5\rangle & &
\end{array}$$

The componentwise form of the condition can be obtained by chasing an arbitrary element $(f, g, h) \in 1 \times 2 \times 3$ around. It can also be obtained directly from the corresponding condition for bicategories (see (4.1) (M.1) in [1]), by putting in further indices in a meaningful way. Starting with $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$, $x \in |F|X$, $y \in |F|Y$, $z \in |F|Z$, $w \in |F|W$, and $\bar{1}, \ldots, \bar{7}$ and $1, \ldots, 10$ from appropriate specification-sets, we use the abbreviations

$$\bar{F}f = F_{x,y;\bar{1}}(f), \qquad \bar{F}g = F_{y,z;\bar{2}}(g), \qquad \bar{F}h = F_{z,w;\bar{4}}(h),$$

$$\bar{F}(g \circ_2 f) = F_{x,z,\bar{3}}(g \circ_2 f), \qquad \bar{F}(h \circ_6 g) = F_{y,w;\bar{5}}(h \circ_6 g),$$

$$\bar{F}(h \circ_9 (g \circ_2 f)) = F_{x,w;\bar{6}}(h \circ_9 (g \circ_2 f)),$$

$$\bar{F}((h \circ_6 g) \circ_{10} f)) = F_{x,w;\bar{7}}((h \circ_6 g) \circ_{10} f)),$$

$$\bar{F}_{1,2} = F_{x,y,z;\bar{1},\bar{2},\bar{3};1,2}, \qquad \bar{F}_{5,6} = F_{y,z,w;\bar{2},\bar{4},\bar{5};5,6},$$

$$\bar{F}_{4,9} = F_{x,z,w;\bar{3},\bar{4},\bar{6};4,9}, \qquad \bar{F}_{8,10} = F_{x,y,w;\bar{1},\bar{5},\bar{7};8,10},$$

$$\bar{F}\alpha_{2,9,6,10} = F_{x,w;\bar{6},\bar{7}}(\alpha_{2,9,6,10}).$$

The following diagram has to be commutative:

$$\begin{array}{ccc}
\bar{F}h \circ_3 (\bar{F}g \circ_1 \bar{F}f) & \xrightarrow{\alpha_{1,3,5,7}} & (\bar{F}h \circ_5 \bar{F}g) \circ_7 \bar{F}f \\
{\scriptstyle \bar{F}h \circ_{3,4} \bar{F}_{1,2}}\downarrow & & \downarrow{\scriptstyle \bar{F}_{5,6} \circ_{7,8} \bar{F}f} \\
\bar{F}h \circ_4 \bar{F}(g \circ_2 f) & \circ & \bar{F}(h \circ_6 g) \circ_8 \bar{F}f \\
{\scriptstyle \bar{F}_{4,9}}\downarrow & & \downarrow{\scriptstyle \bar{F}_{8,10}} \\
\bar{F}(h \circ_9 (g \circ_2 f)) & \xrightarrow[\bar{F}\alpha_{2,9,6,10}]{} & \bar{F}((h \circ_6 g) \circ_{10} f)
\end{array}$$

(vi) (*Identity coherence*) We give the componentwise form of the condition only. For $f: X \to Y$ in $\mathscr{X}$, $x \in |F|X$, $y \in |F|Y$, and appropriate specifications $\bar{1}, \bar{2}, \bar{3}, 1, 2, 3, i, j$,

$$\begin{array}{ccc}
F_{y,y;\bar{1}}(1_{y,i}) \circ_1 F_{x,y;\bar{3}}(f) & \xleftarrow{\;F_{y;\bar{1}} \circ_{1,2} F_{x,y;\bar{3}}(f)\;} & 1_{F_y(Y),j} \circ_2 F_{x,y;\bar{3}}(f) \\
\Big\downarrow{\scriptstyle F_{x,y,y;\bar{3},\bar{1},\bar{2};1,3}} & \circ & \Big\downarrow{\scriptstyle \lambda_{2,j}} \\
F_{x,y;\bar{2}}(1_{y,i} \circ_3 f) & \xrightarrow[\;F_{x,y;\bar{2},\bar{3}}(\lambda_{3,i})\;]{} & F_{x,y;\bar{3}}(f)
\end{array}$$

and a similar commutativity involving the ρ's.

From now on, we assume that the anabicategory $\mathscr{A}$ has saturated identity functors $1_A: \mathbf{1} \to [A, A]$ (1 (iii)), and saturated composition functors $\circ_{A,B,C}: [A,B] \times [B,C] \to [A,C]$ (1 iv)); in this case, let us call $\mathscr{A}$ *saturated*. For $\mathscr{A}$ saturated, there is a useful reformulation of the notion of anafunctor $\mathscr{X} \to \mathscr{A}$, trading in the associativity isomorphisms for certain operations on specifications.

Let $F: \mathscr{X} \to \mathscr{A}$ be an anafunctor of anabicategories. For simplicity, assume that the sets $|F|X$ are singletons. With the notation of 4 (iii), in the context of (3) there, with the subscripts x, y, z dropped, the saturation of the composition functor

$$\circ = \circ_{X,Y,Z}: \mathscr{A}(FX, FY) \circ \mathscr{A}(FY, FZ) \to \mathscr{A}(FX, FZ), \tag{4}$$

gives a uniquely determined $\bar{a} \in |\circ|$ such that $(F_t g) \circ_{\bar{a}} (F_s f) = F_u(g \circ_a f)$ and

$$1_{F_t g} \circ_{b,\bar{a}} 1_{F_s f} = F_{s,t,u;a,b}. \tag{5}$$

Here, $\bar{a}$ depends on (f, g and) s, t, u, a and b; however, the dependence on b is illusory. Consider the following commutative diagrams:

$$\begin{array}{ccccc}
F_t g \circ_{b'} F_s f & & \xrightarrow{\;1_{F_t g} \circ_{b',b} 1_{F_s f}\;} & & F_t g \circ_b F_s f \\
& \searrow{\scriptstyle 1_{F_t g} \circ_{b',\bar{a}} 1_{F_s}} & \circ & \swarrow{\scriptstyle 1_{F_t g} \circ_{b,\bar{a}} 1_{F_s f}} & \\
& & F_t g \circ_{\bar{a}} F_s f & &
\end{array}$$

$$\begin{array}{ccccc}
F_t g \circ_{b'} F_s f & & \xrightarrow{\;1_{F_t g} \circ_{b',b} 1_{F_s f}\;} & & F_t g \circ_b F_s f \\
& \searrow{\scriptstyle F_{s,t,u;\bar{a},b'}} & \circ & \swarrow{\scriptstyle F_{s,t,u,a,b}} & \\
& & F_u(g \circ_{\bar{a}} f) & &
\end{array}$$

(the first is the functoriality of (4), the second is the naturality of $F_{X,Y,Z}$ of 4 (iii)). Together they show that if $\bar{a}$ is suitable for b, then it is suitable for b', with the other data being the same. Thus, we have $\bar{a}$ as a function of s, t, a and u:

4. (iii*) (*Action on specifications*)

$$\bar{a} = F(s,t,u;a) \quad (s \in |F|f,\ t \in |F|g,\ a \in |\circ|((f,g)),$$
$$u \in |F|(g \circ_a f);\ \bar{a} \in |\circ|((F_s f, F_t g)))$$

such that

$$F_u(g \circ_a f) = F_t(g) \circ_{\bar{a}} F_s(f)$$

and

$$F_{u,u'}(\gamma \circ_{a,a'} \varphi) = (F_{t,t'}(\gamma)) \circ_{\bar{a},\bar{a}'} (F_{s,s'}(\varphi))$$

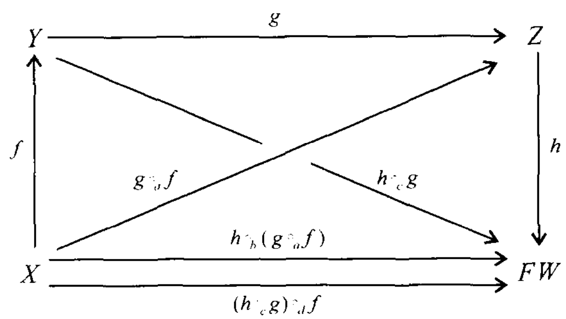
$(\varphi : f \to f', \gamma : g \to g', \bar{a}' = F(s',t',u';a')$; the last equality corresponds to the naturality of $F_{X,Y,Z}$ of 4 (iii)).

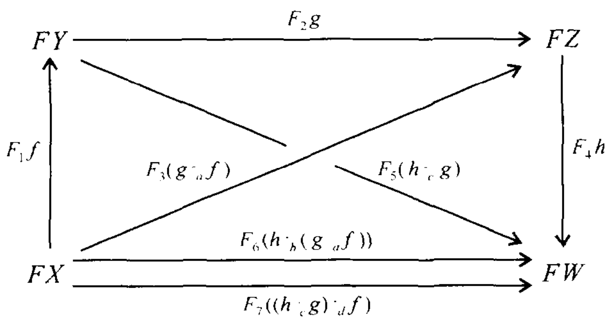
Next, we realize that in this context composition coherence (4 (v)) has as a consequence the identity

4. (v*) (*Action on associativity isomorphisms*)

$$F_{6,7}(\alpha_{a,b,c,d}) = \alpha_{\bar{a},\bar{b},\bar{c},\bar{d}};$$

here reference is made to the data





and $\bar{a} = F(1, 2, 3; a)$, $\bar{b} = F(3, 4, 6; b)$, $\bar{c} = F(2, 4, 5; c)$, $\bar{d} = F(1, 5, 7; d)$.

The identity isomorphisms (4 (iv)) can similarly be traded in for specifications. Let $A \in \mathscr{A}$. The saturation of 1_A means that whenever $\mu: 1_{A,i} \xrightarrow{\cong} \ell$, there is a unique $j \in |1_A|$ such that $\ell = 1_{A,j}$ and $\mu = 1_{A,i,j}$. The identity isomorphism of 4 (iv) has components

$$F_{X,u;i,j}: 1_{FX,j} \xrightarrow{\cong} F_u(1_{X,i}).$$

Given the data in the last display, by the saturation of 1_A, there is unique $\bar{i}$ such that $1_{FX,\bar{i}} = F_u(1_{X,i})$ and

$$1_{FX,j,\bar{i}} = F_{X,u;i,j}. \tag{5'}$$

Again, $\bar{i}$ here does not depend on j; $\bar{i} = F(u; i)$. Thus, the data 4 (iv) can be replaced by

4. (iv*) $\bar{i} = F(u;i) \quad (i \in |\ulcorner 1_X \urcorner|, u \in |F|(1_{X,i}))$

such that

$$F_u(1_{X,i}) = 1_{FX,\bar{i}} \quad \text{and} \quad F_{u,u'}(1_{X,i,i'}) = 1_{FX,\bar{i},\bar{i}'}$$

$(\bar{i} = F(u;i), \bar{i}' = F(u';i'))$.

Then 4 (vi) becomes

4. (vi*) $F_{t,u}(\lambda_{a,i}) = \lambda_{\bar{a},\bar{i}}$ $(f: X \to Y,\ \lambda_{a,i}: 1_{Y,i} \circ_a f \xrightarrow{\cong} f,\ s \in |F| f,\ t \in |F_{Y,Y}|(1_{Y,i}),$ $u \in |F|(1_{Y,i} \circ_a f)$; $\bar{a} = F(s, t, u; a)$, the latter from 4 (iii*), and $\bar{i} = F(t; i)$), and similarly for ρ.

Conversely, even without assuming that $\mathscr{A}$ is saturated, in the definition of an anafunctor $F: \mathscr{X} \to \mathscr{A}$ with all $|F| X$ singletons, we can dispense with the composition isomorphisms 4 (iii) and the identity isomorphisms 4 (iv), and instead, use the data 4 (iii*), 4 (iv*), satisfying 4 (v*) and 4 (vi*). Indeed, if F is given by 4 (i), (ii), (ii*), (iv*), (v*) and (vi*), there is a unique anafunctor $F: \mathscr{X} \to \mathscr{A}$ in the original sense that is related to the F we start with as just described; we use (5) and (5′) as the definition of composition isomorphisms and identity isomorphisms, respectively. The essential point is that 4 (v) and 4 (vi) and will hold.

An *anamonoidal functor* is, by definition, an anafunctor between anabicategories with one object, with the one specification set for the object-function being a singleton. Explicitly, with $\boldsymbol{X}$ and $\boldsymbol{A}$ anamonoidal categories, $\boldsymbol{A}$ saturated, an anamonoidal functor $F: \boldsymbol{X} \to \boldsymbol{A}$ is given by an anafunctor $F: \boldsymbol{X} \to \boldsymbol{A}$ of the ordinary categories $\boldsymbol{X}$ and $\boldsymbol{A}$, a mapping assigning to each $s \in |F| X$, $t \in |F| Y$, $a \in |\otimes|\ (X, Y)$, $u \in |F|$ $(A \otimes_a B)$ a specification $\bar{a} = F(s, t, u; a) \in |\otimes|\ (F_s X, F_t Y)$ such that

$$F_u(X \otimes_a Y) = (F_s X) \otimes_{\bar{a}} (F_t Y),$$

$$F_{u,u'}(f \otimes_{a,a'} g) = (F_{s,s'} f) \otimes_{\bar{a},\bar{a}'} (F_{t,t'} g) \quad (f: X \to X',\ g: Y \to Y', \text{etc.}),$$

and a mapping assigning to each $i \in |I_X|$, $u \in |F|\ (I_{X,i})$, a specification $\bar{i} = F(u; i)$ such that

$$F(I_{X,i}) = I_{A,\bar{i}}$$

$$F(I_{X,i,j}) = I_{A,\bar{i},\bar{j}}$$

and such that the conditions 4 (v*), 4 (vi*) hold.

In case F is a functor (each $|F|\,X$ is a singleton), $\bar{a}$ becomes a function of a alone, thus F is given with a function also denoted $F : |\otimes|\ (X, Y) \to |\otimes|\ (FX, FY)$ and another function $F : |I_X| \to |I_A|$ such that

$$F(X \otimes_a Y) = (FX) \otimes_{Fa} (FY),$$

$$F(f \otimes_{a,a'} g) = (Ff) \otimes_{Fa, Fa'} (Fg) \quad (f : X \to X',\ g : Y \to Y', \text{ etc.}),$$

$$F(\alpha_{a,b,c,d}) = \alpha_{Fa, Fb, Fc, Fd},$$

$$F(\lambda_{a,i}) = \lambda_{Fa, Fi}, \quad F(\rho_{a,i}) = \rho_{Fa, Fi} \quad (\lambda_{a,i} : I_{X,i} \otimes_a A \xrightarrow{\cong} A, \text{ etc.}).$$

We may call such an F a *monoidal functor* between the anamonoidal categories; a monoidal functor is precisely a structure preserving map of anamonoidal categories. It is pointed out next that monoidal functors of anamonoidal categories correspond *exactly* to the (not necessarily strict) monoidal functors of monoidal categories in the usual sense [4].

First, given a monoidal category $\boldsymbol{X}$ is the usual sense, we can construct its *saturation* $\boldsymbol{X}^{\#}$, a saturated anamonoidal category, as follows. The underlying category of $\boldsymbol{X}^{\#}$ is the same as that of $\boldsymbol{X}$; the anaoperation $\otimes^{\#} : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$ is the saturation of the functor $\otimes : \boldsymbol{X} \times \boldsymbol{X} \to \boldsymbol{X}$. This means that $|\otimes^{\#}|\ (X, Y)$ is the class of all entities $a = (X, Y, \mu : X \otimes Y \xrightarrow{\cong} Z)$, with $X \otimes_a Y = Z$, and for $f : X \to X'$, $g : Y \to Y'$, $f \otimes_{a,a'} g = \mu' \circ (f \otimes g) \circ \mu^{-1}$. Similarly, the unit anaobject of $\boldsymbol{X}^{\#}$ is the saturation of $\ulcorner I \urcorner : \mathbf{1} \to \boldsymbol{X}$.

Now, given the monoidal categories $\boldsymbol{X}$ and $\boldsymbol{A}$, we can form their saturations $\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#}$, and we have a *bijection*

$$\mathrm{Hom}(\boldsymbol{X}, \boldsymbol{A}) \xrightarrow{\cong} \mathrm{Hom}(\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#}), \tag{5''}$$

where the first Hom is the collection of all monoidal functors $\boldsymbol{X} \to \boldsymbol{A}$ in the usual sense, the second all monoidal functors $\boldsymbol{X}^{\#} \to \boldsymbol{A}^{\#}$ as defined above. This bijection is given as follows: to $F : \boldsymbol{X} \to \boldsymbol{A}$ corresponds $F^{\#} : \boldsymbol{X}^{\#} \to \boldsymbol{A}^{\#}$, where for a as above,

$$F^{\#}(a) = \langle FX,\ FY,\ F(\mu) \circ i_{X,Y} : FX \otimes FY \xrightarrow{\cong} FZ \rangle,$$

where $i_{X,Y} : FX \otimes FY \xrightarrow{\cong} F(X \otimes Y)$ is the canonical isomorphism given with F; the action of $F^{\#}$ on identity specifications is defined similarly. The verification of the bijection (5″) is omitted (see also [14]).

2. Example (*continued*). Let $\varphi: R \to S$ be a homomorphism of commutative rings. With Mod_R, Mod_S understood as "the" usual monoidal categories, it is common knowledge that we have a monoidal functor

$$F = (\ )\otimes_R S : \mathrm{Mod}_R \to \mathrm{Mod}_S; \tag{6}$$

this is because

(*) $(X \otimes_R S) \otimes_S (Y \otimes_R S)$ is canonically isomorphic to $(X \otimes_R Y) \otimes_R S$ $(X, Y \in \mathrm{MOd}_R)$.

However, in (6) there are two places where non-canonical choices have been made: one is the definition of the monoidal categories, the other is the definition of the functor itself. Making the categories anamonoidal categories in the way explained before, and making the functor an anamonoidal functor, we avoid all non-canonical choices.

Explicitly, F as an anafunctor $F: \mathrm{Mod}_R \to \mathrm{Mod}_S$ is given by $|F|X \underset{\mathrm{def}}{=}$ the class of all universal R-bilinear maps of the form $(X, S) \to Y$; for s the last-displayed map, $F_S(X)$ is Y as an S-module (in the usual sense); the action of F on morphisms is the straightforward one. The action 4 (iii*) on specifications is obtained by making the proof of (*) explicit. Given universal R-bilinear maps

$$(X, S) \xrightarrow{1} \bar{X}, \qquad (Y, S) \xrightarrow{2} \bar{Y}, \qquad (X, Y) \xrightarrow{a} Z, \qquad (Z, S) \xrightarrow{3} \bar{Z},$$

We construct a canonically defined S-bilinear map

$$(\bar{X}, \bar{Y}) \xrightarrow{\bar{a}} \bar{Z}, \tag{7}$$

briefly described as follows. The composite

$$(X, S, Y, S) \to (Z, S) \xrightarrow{3} \bar{Z}$$

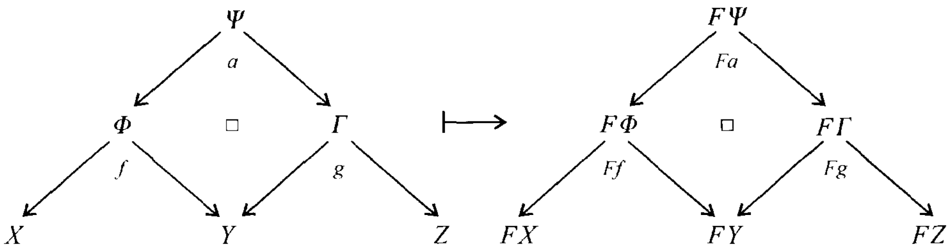
$$(x, u, y, v) \mapsto (a(x, y), u \cdot v) \mapsto 3(a(x, y), u \cdot v)$$

is R-linear in each variable. By the universal property of 1, this gives rise to the componentwise R-linear map

$$(\bar{X}, Y, S) \to \bar{Z},$$

and that of 2 then gives (7) canonically; we can then show that (7) has the required universal property. This is the description of the action 4 (iii*) of F on specifications. The rest of the data and conditions are left to the reader to provide.

3. Example (*continued*). Let $F: \mathbf{X} \to \mathbf{A}$ be a pullback-preserving functor between categories with pullbacks. F gives rise to an anafunctor $\hat{F}: \mathrm{Span}(\mathbf{X}) \to \mathrm{Span}(\mathbf{A})$ of anabicategories. The action of $\hat{F}$ on objects is that of F; the action on arrows of $\hat{F}$ is induced by that of F; now, all specification-sets $|\hat{F}|X$ are singletons, $|\hat{F}|X = \{X\}$, and the effect of F on arrows, $F_{X,Y}: \mathrm{Span}(\mathbf{X})(X, Y) \to \mathrm{Span}(\mathbf{A})(FX, FY)$, is an ordinary functor, for each $X, Y \in \mathbf{X}$. The identity isomorphisms are identities. Instead of the composition isomorphisms, we give the action 4 (iii*) of $\hat{F}$ on specifications; remember there are no s, t, u now. Indeed, this is nothing but the action of F again; $a \in |\circ|(f, g)$ is a pullback square, $\bar{a}$ is its F-image:

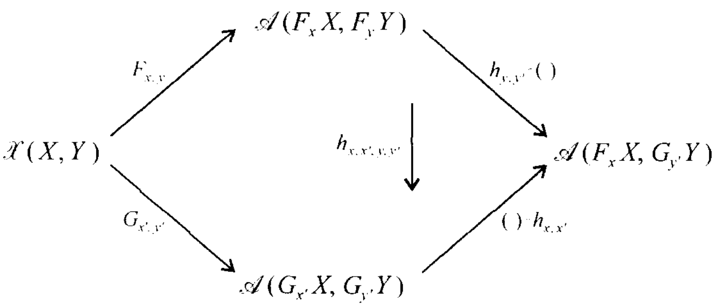


It is immediate that we have 4 (v*): the diagram (2) is mapped into a similar diagram by F.

A *transformation* $h: F \to G$, of anafunctors $\mathscr{X} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{A}$ of anabicategories, is given by data 5 (i), (ii) and condition 5 (iii), (iv).

5. (i) For $X \in \mathscr{X}$, $x \in |F|X$, $x' \in |G|X$, an anaobject $h_{x,x'}$ of the hom-category $\mathscr{A}(F_x X, G_{x'} X)$, $h_{x,x'}: \mathbf{1} \xrightarrow{\text{a}} \mathscr{A}(F_x X, G_{x'} X)$. $h_{x,x'}$ has components $h_{x,x',m}: F_x X \to G_{x'} X$ $(m \in |h_{x,x'}|)$.

(ii) For the data in (i) and $Y \in \mathscr{X}$, $y \in |F|Y$, $y' \in |G|Y$, a natural transformation $h_{x,x',y,y'}$ as in



Here, $h_{y,y'} \circ (\ )$ abbreviates the composite of

$$\mathscr{A}(F_x X, F_y Y) \cong \mathbf{1} \times \mathscr{A}(F_x X, F_y Y) \xrightarrow{h_{y,y'} \times \cdots} \mathscr{A}(F_y Y, G_{y'} Y) \times \mathscr{A}(F_x X, F_y Y)$$

$$\longrightarrow \mathscr{A}(F_x X, G_{y'} Y)$$

$h_{x,x',y,y'}$ has components

$$h_{x,x',y,y':s,s':a,b;m,n}:$$

$$h_{y,y':n} \circ_a F_{x,y:s}(f) \to G_{x',y':s'}(f) \circ_b h_{x,x':m}.$$

The conditions that follow will be given in their componentwise forms only.

(iii) With reference to

$$X \xrightarrow{f} Y \xrightarrow{g} Z, X \xrightarrow{g\circ_1 f} Y,$$

$$\bar F f = F_{x,y;s}(f), \qquad \bar F g = F_{y,z;t}(g), \qquad \bar F(g\circ_1 f) = F_{x,z;u}(g\circ_1 f),$$

$$\bar G f = G_{x',y';s'}(f), \qquad \bar G g = G_{y',z';t'}(g), \qquad \bar G(g\circ_1 f) = G_{x',z';u'}(g\circ_1 f),$$

$$\bar F_{1,4} = F_{x,y,z;s,t,u;1,4}, \qquad \bar G_{1,6} = G_{x',y',z';s',t',u';1,6},$$

$$\bar h_X = h_{x,x',m}, \qquad \bar h_Y = h_{y,y',n}, \qquad \bar h_Z = h_{z,z',p},$$

$$h_{2,3} = h_{x,x',z,z';u,u';2,3;m,p}, \qquad h_{8,12} = h_{y,y',z,z';t,t';8,12;n,p},$$

$$h_{14,10} = h_{x,x',y,y';s,s';14,10;m,n},$$

$$\alpha 1 = \alpha_{4,5,8,9}, \qquad \alpha 2 = \alpha_{10,11,6,7}, \qquad \alpha 3 = \alpha_{14,15,12,13},$$

$$\begin{array}{ccc}
\bar h_Z\circ_2\bar F(g\circ_1 f) & \xrightarrow{\quad h_{2,3}\quad} & \bar G(g\circ_1 f)\circ_3\bar h_X \\
{\scriptstyle \bar h_Z\circ_{5,2}\bar F_{1,4}}\uparrow\cong & & \cong\uparrow{\scriptstyle \bar G_{6,1}\circ_{7,3}\bar h_X} \\
\bar h_Z\circ_5(\bar Fg\circ_4\bar Ff) & \circ & (\bar Gg\circ_6\bar Gf)\circ_7\bar h_X \\
{\scriptstyle \alpha 1}\downarrow\cong & & \cong\uparrow{\scriptstyle \alpha 2} \\
(\bar h_Z\circ_8\bar Fg)\circ_9\bar Ff & & \bar Gg\circ_{11}(\bar Gf\circ_{10}\bar h_X) \\
{\scriptstyle h_{8,12}\circ_{9,13}\bar Ff}\downarrow & & \uparrow{\scriptstyle \bar Gg\circ_{15,11}h_{14,10}} \\
(\bar Gg\circ_{12}\bar h_Y)\circ_{13}\bar Ff & \xleftarrow[\alpha 3]{\cong} & \bar Gg\circ_{15}(\bar h_Y\circ_{14}\bar Ff)
\end{array}$$

(iv) With all appropriate values of the parameters,

$$\begin{array}{ccccc}
h_{x,x';m}\circ_1 1_{F_X X,j} & \xrightarrow[\cong]{\rho_{1,j}} & h_{x,x'} & \xleftarrow[\cong]{\lambda_{2,k}} & 1_{G_{X'} X,k}\circ_2 h_{x,x';m} \\
\uparrow{\scriptstyle h_{x,x'}\circ_{3,1}F_{x;s}} & & \circ & & {\scriptstyle G_{x';t}\circ_{4,2}h_{x,x'}}\uparrow \\
h_{x,x';m}\circ_3 F_{x,x;s}1_{X,i} & & \xrightarrow[h_{x,x',x,x';s,t;3,4;m,m}]{} & & G_{x',x';t}1_{X,i}\circ_4 h_{x,x';m}
\end{array}$$

The transformation h is *natural* (the more usual terminology would be: *strong*) if each natural transformation $h_{x,x',y,y'}$ is an isomorphism.

Consider saturated anamonoidal categories and anamonoidal functors between them: $X \overset{F}{\underset{G}{\rightrightarrows}} A$. A transformation $h: F \to G$ (in the sense of one between anafunctors of anabicategories) involves an anaobject $h_{*,*}$ of A. Let us take this to be I_A, the unit anaobject of A; let us call such transformations h *monoidal*. Then

$$\bar{h} \underset{\text{def}}{=} h_{s,s';a,b;m,m} : F_S(X) \otimes_a I_{A,m} \to I_{A,m} \otimes_b G_{s'}(X)$$

gives rise, via the identity isomorphisms, to $h_{s,s'} : F_S(X) \to G_{s'}(X)$:

$$\begin{array}{ccc} F_s(X)\otimes_a I_{A,m} & \xrightarrow{\quad \bar{h} \quad} & I_{A,m}\otimes_b G_{s'}(X) \\ {\scriptstyle \rho_{a;m}}\downarrow \cong & \circ & \cong \downarrow {\scriptstyle \lambda_{b;m}} \\ F_s(X) & \xrightarrow[\quad h_{s,s'} \quad]{} & G_{s'}(X) \end{array}$$

indeed, we easily see that the resulting lower horizontal arrow is independent of the parameters a, b and m. Inspection shows that conditions 5 (iii) and (iv) reduce to the following:

$$\begin{array}{ccc} F_u(X\otimes_a Y) & \xrightarrow{\quad h_{u,u'} \quad} & G_{u'}(X\otimes_a Y) \\ \Vert & \Vert & \Vert \\ F_s X\otimes_{\bar{a}} F_t Y & \xrightarrow[\quad h_{s,s'}\otimes_{\bar{a},\hat{a}} h_{t,t'} \quad]{} & G_{s'} X\otimes_{\hat{a}} G_{t'} Y \end{array} \quad , \quad \text{i.e., } h_{u,u'} = h_{s,s'} \otimes_{\bar{a},\hat{a}} h_{t,t'},$$

with $\bar{a} = F(s, t, u; a)$ and $\hat{a} = G(s', t', u'; a)$; and

$$h_{u,v} = 1_{A,F(u;i),G(v;i)} (: I_{A,F(u;i)} \to I_{A,G(v;i)})$$

(Here $i \in |I_X|, u \in |F|(I_{X,i}), v \in |G|(I_{X,i})$; recall that $F(u; i), G(v; i) \in |I_A|$.)

When F and G are (monoidal) functors, these data and conditions further reduce to

$$h_A : FA \to GA,$$

$$h_{A \otimes_a B} = h_A \otimes_{Fa,Ga} h_B,$$

$$h_{I_{X,i}} = 1_{A,Fi,Gi}.$$

Now, the mapping (5″) becomes an *isomorphism of categories*

$$\mathrm{Hom}(\boldsymbol{X}, \boldsymbol{A}) \xrightarrow{\cong} \mathrm{Hom}(\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#})$$

between the category $\mathrm{Hom}(\boldsymbol{X}, \boldsymbol{A})$ of monoidal functors $\boldsymbol{X} \to \boldsymbol{A}$ and monoidal natural transformations in the classical sense (see [4]), and the category $\mathrm{Hom}(\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#})$ of the monoidal functors between the saturations $\boldsymbol{X}^{\#}, \boldsymbol{A}^{\#}$ and their monoidal transformations.

4. The anabicategory of saturated anafunctors

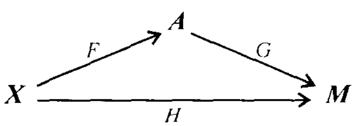
The saturated composition of two saturated anafunctors (or sanafunctors, for short) is obtained by saturating the ordinary composite introduced in Section 1. Since saturation involves taking quotients of equivalence relations, it is natural to define the composite of sanafunctors up to isomorphism only, by allowing the quotients to be replaced by sets that are bijectively related to them. In this way, we get the meta-*ana*bicategory SANACAT$^{\#}$ of categories and sanafunctors, and the anabicategory SanaCat$^{\#}$ of small categories and (small) sanafunctors between them. From the point of view of the present paper, SanaCat$^{\#}$ and SANACAT$^{\#}$ seem to be the right universes for category theory.

We now assemble the definition of SanaCat$^{\#}$, the anabicategory of small categories and saturated anafunctors, in explicit terms; the consideration of SANACAT$^{\#}$ will be left to the reader.

Recall that, for $\boldsymbol{X}, \boldsymbol{A}$ small categories, Sana $(\boldsymbol{X},\boldsymbol{A})$ denotes the category of sanafunctors from $\boldsymbol{X}$ to $\boldsymbol{A}$ and their natural transformations. With $\boldsymbol{X}, \boldsymbol{A}$ and $\boldsymbol{M}$ arbitrary small categories, we define the *composition anafunctor*

$$\circ = \circ_{\boldsymbol{X},\boldsymbol{A},\boldsymbol{M}} : \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A}) \times \mathrm{Sana}(\boldsymbol{A},\boldsymbol{M}) \to \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M}) \tag{1}$$

as follows. Recall the criterion for a "bi-anafunctor" stated in Section 1. after 1.13; according to this, we only have to define, for $F \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G \in \mathrm{Sana}(\boldsymbol{A},\boldsymbol{M})$ and $H \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M})$, that is,



(2)

the specification-sets $|\circ|\,((F,G),H)$ and the sections

$$(\ )\circ F : \mathrm{Sana}(\boldsymbol{A},\boldsymbol{M}) \to \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M}), \tag{3}$$

$$G\circ(\ ) : \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M}), \tag{4}$$

appropriately.

For $F \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G \in \mathrm{Sana}(\boldsymbol{A},\boldsymbol{M})$ and $H \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{M})$, $|\circ|\,((F,G),H)$ is defined as the set of all families $\alpha = \langle \alpha_X \rangle_{X \in \mathrm{Ob}(\boldsymbol{X})}$ of maps

$$\alpha_X \in \prod_{(s \in |F|(X),\, t \in |G|(F_s X))} |H|(X, G_t F_s X)$$

$$\alpha_X : (s \in |F|(X), t \in |G|(F_S X)) \mapsto \alpha_X(s,t) \in |H|(X, G_t F_s X). \tag{5}$$

such that α satisfies the following condition (i).

(i) For all $f: X \to Y$, $s \in |F|(X)$, $t \in |G|(F_S X)$, $u \in |F|(Y)$, $v \in |G|(F_S Y)$, if $a = \alpha_X(s,t)$, $b = \alpha_Y(u,v)$, we have

$$G_{t,v}(F_{s,u}(f)) = H_{a,b}(f).$$

α as in (5) satisfying (i) is called a *composition specification* (*for* "$H = F\circ G$").

Note that, under the notation of (i), $H_a(X) = (G \circ_\alpha F)_a(X) = G_t F_s(X)$ when $\alpha_X(s,t) = a$.

We have that any composition specification α for "$H = F \circ G$" satisfies the following (ii) and (iii).

(ii) *For any* $X \in \mathrm{Ob}(\boldsymbol{X})$, $M \in \mathrm{Ob}(\boldsymbol{M})$, $a \in |H|(X,M)$ *and* $s \in |F|X$, *there is* $t \in |G|(F_S X)$ *such that* $M = G_t F_s X$ *and* $\alpha_X(s,t) = a$.
Indeed, take any $v \in |G|(F_S X)$, and let $b = \alpha_X(s,v)$; since G is saturated, there is a (unique) $t \in |G|(F_S X, M)$ such that

$$G_{v,t}(1_{F_s X}) = H_{b,a}(1_X) : G_v F_s X \to M.$$

But then for $a' = \alpha_X(s,t)$, by (i) we have $H_{b,a}(1_X) = H_{b,a'}(1_X)$, which, since H is saturated, implies $a' = a$.

(iii) *For all* X, M, s, t, u, v *such that* $G_t F_s X = G_v F_u X = M$ *we have that* $\alpha_X(s,t) = \alpha_X(u,v)$ *iff* $G_{t,v} F_{s,u}(1_X) = 1_M$.

Indeed, by (ii), $G_{t,v} F_{s,u}(1_X) = H_{ab}(1_X)$ with $a = \alpha_X(s,t)$, $b = \alpha_X(u,v)$; since H is saturated, $H_{ab}(1_X) = 1_M$ iff $a = b$.

A *partial composition specification for* "$H = G \circ F$" is a family $\alpha = \langle \alpha_X \rangle_{X \in \mathrm{Ob}(\boldsymbol{X})}$ of functions α_X with domain $\mathrm{dom}(\alpha_X) \subset \coprod_{s \in |F|X} |G|(F_s X) = \{(s,t){:}\ s \in |F|(X),\ t \in |G|(F_s X)\}$ such that for any $X \in \mathrm{Ob}(\boldsymbol{X})$, there are $s \in |F|(X)$, $t \in |G|(F_s X)$ with $(s,t) \in \mathrm{dom}(\alpha_X)$, and (i) holds for $(s,t) \in \mathrm{dom}(\alpha_X)$, $(u,v) \in \mathrm{dom}(\alpha_Y)$. Any partial composition specification can be uniquely extended to a complete one for the same relation "$H = G \circ F$"; given any $u \in |F|(X)$, $v \in |G|(F_u X)$, choose (s,t) as above, and define $b = \alpha_X(u,v)$ so as to satisfy

$$G_{t,v}(F_{s,u}(1_X)) = H_{a,b}(1_X);$$

a unique such b exists because H is saturated. It is easy to verify that the full specification so defined, also denoted by α, will satisfy (i).

To define the section (3), first of all, we have to show that $|\circ|\,((F,G))$ is inhabited (F and G as before). For $H = G \circ_\alpha F$, with suitable α, we take $(G \circ F)^{\#}$; here $G \circ F$ is the composite as anafunctors in the sense of Section 1, and $(\ )^{\#}$ is the saturation functor. According to the definition of $H = (G \circ F)^{\#}$, a typical element of $|H|\,(X,M)$ is an equivalence class $[(s,t,\mu)]$ with $s \in |F|(X)$, $t \in |G|(F_s X)$ and $\mu : G_t F_s X \xrightarrow{\cong} M$, where the underlying equivalence relation is defined by

$$(s,t,\mu) \sim (s',t',\mu') \iff \begin{array}{ccc} G_t F_s X & \xrightarrow{G_{tt'} F_{ss'} 1_X} & G_{t'} F_{s'} X \\ & {\scriptstyle \mu} \searrow \quad \circ \quad \swarrow {\scriptstyle \mu'} & \\ & M & \end{array}$$

To define the composition specification α, we put $\alpha_X(s,t) = [(s,t,1_{G_t F_s X})]$ ($s \in |F|X$, $t \in |G|(F_s X)$). With further data as in (i), the definition of $H = (G \circ F)^{\#}$ gives that $H_{[(s,t,1)],[(u,v,1)]}(f) = G_{t,v} F_{s,u}(f)$; thus, (i) indeed holds. We have shown that $|\circ|\,((F,G))$ is inhabited.

To define the effect of the section (3) on arrows, we take

$$X \xrightarrow{F} A \underset{G'}{\overset{G}{\underset{\Downarrow \nu}{\rightrightarrows}}} M$$

and for $\alpha \in |\circ|((F,G))$, $\alpha' \in |\circ|((F,G'))$, we define

$$\nu \circ_{\alpha,\alpha'} F : (G \circ_\alpha F) \to (G' \circ_{\alpha'} F); \tag{6}$$

that is, we determine the component

$$(\nu \circ_{\alpha,\alpha'} F)_{a,a'} : (G \circ_\alpha F)_a X \to (G' \circ_{\alpha'} F)_{a'} X$$

for any $a \in |G \circ_\alpha F|X$, $a' \in |G' \circ_{\alpha'} F|X$. But, by (ii), we can take $s \in |F|X$, $t \in |G|(F_s X)$, $t' \in |G'|(F_s X)$, such that $\alpha_X(s,t) = a$, $\alpha_X(s,t') = a'$; we define

$$(\nu \circ_{\alpha,\alpha'} F)_{a,a'} = \nu_{t,t'} : {}_t G_t F_s X \to G_{t'} F_s X. \tag{7}$$

If, instead of s, t, t', we take u, v, v', then by (ii), and the commutativity of

$$\begin{array}{ccc}
G_t F_s X & \xrightarrow{\nu_{t,t'}} & G_{t'} F_s X \\
{\scriptstyle G_{t,v}(F_{s,u} 1_X)} \downarrow \; {\scriptstyle = 1_{H_a}} & & {\scriptstyle 1_{H_{a'}} =} \downarrow \; {\scriptstyle G_{t',v'}(F_{s,u} 1_X)} \\
G_v F_u X & \xrightarrow[\nu_{v,v'}]{} & G_{v'} F_u X
\end{array}$$

(the naturality of ν), we see that the same value is obtained in (7). We omit the verification that (6) so defined is a natural transformation, and that (3) so defined is an anafunctor.

The definition of (4) is analogous; we get, for

$$X \underset{F'}{\overset{F}{\underset{\Downarrow \rho}{\rightrightarrows}}} A \xrightarrow{G} N,$$

$$(G \circ_{\alpha,\beta''})_{a,b} = G_{t,u}(\rho_{s,s'}) : G_t F_s X \longrightarrow G_u F_{s'} X$$

($\alpha_X(s,t) = a$, $\alpha_X(s',u) = b$). Finally, condition (ii) in the characterization of "bi" anafunctors is

$$\begin{array}{ccc}
G \circ_\alpha F & \xrightarrow{G \circ_{\alpha,\beta} \rho} & G \circ_\beta F' \\
{\scriptstyle \nu \circ_{\alpha,\alpha'} F} \downarrow & \circ & \downarrow {\scriptstyle \nu \circ_{\beta,\beta'} F'} \\
G' \circ_{\alpha'} F & \xrightarrow[G' \circ_{\alpha',\beta'} \rho]{} & G' \circ_{\beta'} F'
\end{array}$$

which becomes, in components,

$$\begin{array}{ccc} G_t F_s X & \xrightarrow{G_{tu}\rho_{ss'}} & G_u F'_{s'} X \\ {\scriptstyle v_{tt'}}\downarrow & \circ & \downarrow{\scriptstyle v_{uu'}} \\ G_{t'} F_s X & \xrightarrow[G_{t'u'}\rho_{ss'}]{} & G_{u'} F'_{s'} X \end{array}$$

which holds by the naturality of v.

In fact, the composition anafunctor (1) is itself saturated. To see this, let $H = G \circ_\alpha F$ and $h: H \xrightarrow{\cong} K$; by 1.9, h is given by bijections

$$(a \mapsto \bar{a}): |G \circ_\alpha F|(X, M) \xrightarrow{\cong} |K|(X, M).$$

We want to show that there is a unique $\beta \in |\circ|((F, G), K)$ for which $h = 1_G \circ_{\alpha,\beta} 1_F$. For $X \in \mathbf{X}$, $s \in |F|X$, $t \in |G|(F_s X)$, and $M = G_t F_s X$, let us define $\beta_X(s,t) \underset{\text{def}}{=} \overline{\alpha_X(s,t)}$. We have that β satisfies (i): with $\alpha_X(s,t) = a$, $\alpha_X(u,v) = b$, we have $G_{tv} F_{su}(f) = H_{ab}(f)$ since α is a composition specification; but $H_{ab}(f) = K_{\bar{a}\bar{b}}(f)$ by 1.9; the desired equality follows.

Let $a \in |G \circ_\alpha F|\ (X, M)$. On the one hand, $h_{a,\bar{a}} = 1_M$ (see 1.9). On the other hand, choose (s,t) such that $\alpha_X(s,t) = a$; then $\beta_X(s,t) = \bar{a}$, and $(1_G \circ_{\alpha,\beta} 1_F)_{a,\bar{a}} = (1_G \circ_{\alpha,\beta} F)_{a,\bar{a}} = (1_G)_{t,t} = 1_M$. This shows that $h_{a,\bar{a}} = (1_G \circ_{\alpha,\beta} 1_F)_{a,\bar{a}}$; since we have "enough" pairs of specifications $(a, \bar{a})$ (see 1.7), it follows that $h = 1_G \circ_{\alpha,\beta} 1_F$. It is also clear that β is uniquely determined.

For any small category $\mathbf{X}$, the *identity sanafunctor* $1^{\#}_{\mathbf{X}}: \mathbf{X} \to \mathbf{X}$, the saturation of the ordinary identity functor, has $|1^{\#}_{\mathbf{X}}|\ (X, X') = \mathrm{Iso}(X, X')$, the set of all isomorphisms from X to X', and, for $f: X \to Y$, $i \in |1^{\#}_{\mathbf{X}}|\ (X, X')$, $j \in |1^{\#}_{\mathbf{X}}|\ (Y, Y')$,

$$(1^{\#}_{\mathbf{X}})_{i,j}(f) = j \circ f \circ i^{-1}: X' \to Y'.$$

The identity anaobjects in SanaCat$^{\#}$ are given by the ordinary objects $1^{\#}_{\mathbf{X}}$.

We may make SanaCat$^{\#}$ into a saturated anabicategory, by taking the saturation $\ulcorner 1^{\#}_{\mathbf{X}} \urcorner^{\#}: \mathbf{1} \xrightarrow{\mathrm{a}} [\mathbf{X}, \mathbf{X}]$ to be the identity anaobject for $\mathbf{X}$; we will not carry out this change.

For $F: \mathbf{X} \to \mathbf{A}$, the component $\lambda_\alpha: 1^{\#}_{\mathbf{A}} \circ_\alpha F \xrightarrow{\cong} F$ of the left identity isomorphism $\lambda = \lambda_{\mathbf{X},\mathbf{A}}: 1^{\#}_{\mathbf{X}} \circ_\alpha (\ ) \xrightarrow{\cong} 1_{[\mathbf{X},\mathbf{A}]}$ is given (see 1.9) by the mappings

$$\lambda[X, A]: (a \in |1^{\#}_{\mathbf{A}} \circ F|(X, A)) \mapsto (\bar{a} \in |F|(X, A)),$$

one for each pair $(X \in \mathrm{Ob}(\mathbf{X}), A \in \mathrm{Ob}(\mathbf{A}))$, defined thus: if $s \in |F|X$, $i \in |1^{\#}_{\mathbf{A}}|(F_s X)$, $a = \alpha_X(s, i)$, then $\bar{a}$ is determined (through the saturatedeness of F) by the conditions $F_{\bar{a}} X = A$, $F_{s\bar{a}}(1_X) = i$; it is easy to see that this does define $a \mapsto \bar{a}$ uniquely, and the equality (see 1.9) $(1^{\#}_{\mathbf{A}} \circ F)_{a,b}(f) = F_{\bar{a},\bar{b}}(f)$ holds $(f: X \to Y,\ a \in |1^{\#}_{\mathbf{A}} \circ F|(X)$, $b \in |1^{\#}_{\mathbf{A}} \circ F|(Y))$, showing that λ_α is properly defined. The naturality of λ has also to be checked. Similarly, we define the right identity isomorphisms.

The associativity isomorphism

$$\alpha = \alpha_{X,A,M,P} : (\circ_{X,A,P}) \circ (\circ_{X,A,M} \times [M,P]) \xrightarrow{\cong} (\circ_{X,A,P}) \circ ([X,A] \times \circ_{A,M,P})$$

has components

$$\alpha = \alpha_{\alpha,\beta,\gamma,\delta} : H \circ_\beta (G \circ_\alpha F) \xrightarrow{\cong} (H \circ_\gamma G) \circ_\delta F$$

($F \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, $G \in \mathrm{Sana}(\boldsymbol{A},\boldsymbol{M})$, $H \in \mathrm{Sana}(\boldsymbol{M},\boldsymbol{P})$, etc.), defined through the renaming transformation $\bar{\alpha}$ given by the maps

$$\bar{\alpha}[X,P] : (a \in |H \circ_\beta (G \circ_\alpha F)|(X,P)) \overset{\cong}{\longmapsto} (\bar{a} \in |(H \circ_\gamma G) \circ_\delta F)|(X,P))$$

($X \in \mathrm{Ob}(\boldsymbol{X})$, $P \in \mathrm{Ob}(\boldsymbol{P})$) thus: if $a = \beta_X(\alpha_X(s,t),u)$, then $\bar{a} = \delta_X(s, \gamma_{F,X}(t,u))$.

I leave all verifications, including the associativity and identity coherences, to the reader.

Extending 1.10, we have

SanaCat$^{\#}$ *and* AnaCat *are equivalent as anabicategories.*

We have not given the notion of equivalence of anabicategories; the formulation of this notion, and the verification of the last-stated proposition are left to the reader.

5 The effects of weak versions of the axiom of choice

In the previous parts of the paper, we left open whether the bicategory AnaCat is Cartesian closed. In this section, we show that a very weak version of the AC, one that is consistent with the negations of most of the usual special cases of the AC, is sufficient to ensure that the said conclusion holds.

This section is somewhat incomplete; since the first version of this paper was written, further, and partly better, results have been found, in a collaboration of Robert Paré and the author; it is planned that they will be described in [15]. On the other hand, it incorporates substantial improvements that were kindly communicated to me by the referee.

In this section, we sometimes use classical logic; the marking (CL) indicates that the result in question depends on classical logic (the principle of excluded middle).

For sets A, B, $A \cong B$ abbreviates that there is a bijection $A \xrightarrow{\cong} B$. I propose the following axiom of class-set theory.

Small Cardinality Selection Axiom (SCSA). There is a class-function assigning, to each set A, a set $\|A\|$ and a bijection $\iota_A : A \xrightarrow{\cong} \|A\|$ such that, for each set B, the class $\{\|A\| : A \cong B\}$ is a set.

Of course, under the Global AC (there is a class-function that assigns to each inhabited set a member of that set), and by using classical logic (whose validity is a consequence of the AC, by the well-known argument of R. Diaconescu (see [9, 5.23, p. 14]), we have the SCSA in the strong form when $\langle B \rangle = \{\|A\| : A \cong B\}$ is a singleton; now, $\|A\|$ is the usual cardinality of A; the global version of choice is needed for the function $A \mapsto \iota_A$.

As the referee has pointed out, and as will be explained below, the SCSA is related to Blass' Axiom of Small Violations of Choice (SVC) (see [3, Section 4., p. 41]). This axiom is as follows.

SVC. There is a set S such that, for every set A, there exists an ordinal α and a function from a subset of $S \times \alpha$ onto A.

The following global version of the SVC was also pointed out by the referee, although it appeared implicitly in the first version of this paper, as a fact about the universe of sets constructible from a fixed set.

GSVC. There are a set S and a class-function mapping $S \times \mathbf{Ord}$ onto $\mathbf{V}$, the class of all sets.

In [3], Blass shows that the SVC is a "very weak form of the axiom of choice". It holds in $L(T)$, the universe of sets constructible over an arbitrary transitive set T ($T \in L(T)$), and it holds in HOD(T), the universe of hereditarily ordinal definable sets over T, for an arbitrary set T (use T as a single additional parameter, in addition to ordinals, in definitions of members of HOD(T)). Moreover, it holds in P. Cohen's symmetric models, derived from generic models, used by Cohen to show the independence of the AC. Any of these facts show that SVC is consistent with the negation of any of a certain class of special cases of the AC; such a special case, call it AC*, may e.g., be the statement that there is a well-ordering of the set of all reals. If we have a model M of set-theory in which AC* fails, then there is a (transitive) set T "responsible" for this failure (the transitive closure of $\mathbb{R}$ in the example), and $L^{(M)}(T)$, that is, $L(T)$ taken relative to M, will also exhibit the failure of AC*; but $L^{(M)}(T)$ satisfies the SVC.

On the other hand, Blass shows the independence of SVC from ZF by a forcing argument using a proper class of forcing conditions.

All the mentioned results of Blass seem to have straightforward variants for the GSVC; in the first version of this paper, a detailed proof was given of the fact that $L(T)$, with T a transitive set, satisfies the GSVC.

The next result, the fact that the GSVC implies the SCSA, is due to the referee. In the first version of the paper, I had the weaker result that, assuming that the AC holds in V, we have that any standard model of G–B satisfying the GSVC also satisfies the SCSA.

1. (CL) *The GSVC implies the SCSA.*

Proof. Let, by the GVSC, S be a set and $E: S \times \mathbf{Ord} \twoheadrightarrow \mathbf{V}$ a surjection. For any $a \in \mathbf{V}$, let ρ_a be the least ordinal ρ for which $\{s \in S : E(s, \rho) = a\} \neq \emptyset$, and let $j(a) = \{s : s \in S, E(s, \rho_a) = a\}$; clearly, $j(a) \neq \emptyset$.

Let A be an arbitrary set. Define $R_A = \{\rho_a : a \in A\}$, and let $\pi_A : R_A \xrightarrow{\cong} \lambda_A$ be the unique order-preserving bijection of R_A (ordered as the ordinals are) onto an ordinal. Let $\|A\| = \{(j(a), \pi_A(\rho_a)) : a \in A\}$, and $i_A : A \to \|A\|$ be given by $i_A(a) = (j(a), \pi_A(\rho_a))$. Note that $\|A\| \subset S \times \lambda_A$. Clearly, i_A is a surjection. But if $a, b \in A$ and $i_A(a) = i_A(b)$, then $\rho_a = \rho_b$, and picking any $s \in j(a) = j(b)$, we have that $a = E(s, \rho_a) = E(s, \rho_b) = b$; this shows that i_A is also an injection, and thus a bijection, $i_A : A \xrightarrow{\cong} \|A\|$.

Let B be an arbitrary set. The class C of ordinals that can be mapped into $\mathscr{P}(B)$ in a 1–1 way is a set (Hartog's theorem), since the mapping with domain the set $\mathscr{W}$ of all well-orderings $(W, <_W)$ of subsets W of $\mathscr{P}(B)$, assigning to $(W, <_W)$ the order-type of $(W, <_W)$, is a surjection $\mathscr{W} \to C$. Thus, there exist ordinals that cannot be mapped 1–1 into $\mathscr{P}(B)$; let α be any, e.g. the least, such.

I claim that for any A such that $A \cong B$, we have that $\|A\| \subset S \times \alpha$. Once this is shown, we obviously have that $\langle B \rangle = \{\|A\| : A \cong B\}$ is a set as required.

If $f: B \twoheadrightarrow \alpha$ is a surjection, then the map $\xi \mapsto f^{-1}(\xi) : \alpha \to \mathscr{P}(B)$ is 1–1; it follows that B cannot be mapped surjectively onto α. But if $g : B \cong A$, then we have

$$B \xrightarrow[\cong]{g} A \xrightarrow{a \mapsto \rho_a} R \xrightarrow[\cong]{\pi_A} \lambda_A,$$

and if also $\lambda_A \geq \alpha$, then $(\alpha \neq \emptyset)$ λ_A, and hence also B, can be mapped surjectively onto α. It follows that if $A \cong B$, the $\lambda_A < \alpha$. Therefore, if $A \cong B$, then $\|A\| \subset S \times \lambda_A \subset S \times \alpha$ as desired.

As a consequence of 2 and remarks made earlier, we have, e.g., that

2. (CL) *For an arbitrary transitive set T, $L(T)^{\hat{}}$ is a model of* G-B + SCSA. *In fact, the class-function witnessing the* SCSA *can be chosen to be a definable class* (that is, definable with parameters in the structure $(X, \in | X)$ *for* $X = L(T)$).

We turn to the effect of SCSA on anafunctors.

3. (SCSA) (*Assume the* SCSA.) *For any small categories* $\mathbf{X}$, $\mathbf{A}$, *there is a small full subcategory* $\mathrm{A}(\mathbf{X}, \mathbf{A})$ *of* $\mathrm{Sana}(\mathbf{X}, \mathbf{A})$ *such that the inclusion* $\mathrm{A}(\mathbf{X}\mathbf{A}) \to \mathrm{Sana}(\mathbf{X}, \mathbf{A})$ *is an equivalence of categories.*

Proof. We use the notation of the statement of the SCSA; we write $\langle \mathrm{B} \rangle$ for $\{\|A\| : A \cong B\}$. Let Γ be the *set* of all sets $\mathscr{G}$ for which there are a subset S of $\mathrm{Ob}(\mathbf{X}) \times \mathrm{Ob}(\mathbf{A})$ and a function $\Phi \in \prod_{(X,A) \in S} \langle \mathrm{Iso}(A, A) \rangle$ (Iso (A, A) is the set of all isomorphisms $A \xrightarrow{\cong} \mathrm{A}$ in $\mathbf{A}$)) such that $\mathscr{G}$ is of the form

$$\mathscr{G} = \coprod_{(X,A) \in S} \Phi(X, A) \underset{\text{def}}{=} \{((X, A), u) : u \in \Phi(X, A)\};$$

indeed, it is clear that Γ is a set. The class (in fact, set) of objects of $\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ is defined to be the set of sanafunctors (saturated anafunctors) $G:\boldsymbol{X} \to \boldsymbol{A}$ such that $|G|$, the set of specifications of G, belongs to Γ. Since Γ is a set, and $\boldsymbol{X},\boldsymbol{A}$ are small, it is clear that the class of sanafunctors described is a set ; $\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$, a full subcategory of $\mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, is small.

To show that the inclusion $\mathrm{A}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$ is an equivalence, we exhibit, for any $F \in \mathrm{Sana}(\boldsymbol{X},\boldsymbol{A})$, a sanafunctor $[\![F]\!] \in \mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ and an isomorphism $\iota_F: F \xrightarrow{\cong} [\![F]\!]$. Let F be given. Remember that, since F is saturated, $|F|(X,A) \cong \mathrm{Iso}(A,A)$ and hence, $\| |F|(X,A)\| \in \langle \mathrm{Iso}(A,A)\rangle$ whenever $X \in \boldsymbol{X}$, $A \in \boldsymbol{A}$ and $|F|\ (X,A)$ is inhabited (see section 1, (2′)). We put

$$S = \{(X,A) \in \mathrm{Ob}(\boldsymbol{X}) \times \mathrm{Ob}(\boldsymbol{A}) : |F|(X,A) \text{ is inhabited}\},$$

$$|[\![F]\!]| = \coprod_{(X,A) \in S} \| \, |F|(X,A)\|;$$

this ensures that $|[\![F]\!]| \in \Gamma$, and thus, once $[\![F]\!]$ is fully defined, that $[\![F]\!] \in \mathrm{A}(\boldsymbol{X},\boldsymbol{A})$. Note that we have

$$|[\![F]\!]| = \coprod_{X \in \boldsymbol{X},\, A \in \boldsymbol{A}} \| \, |F|(X,A)\|; \tag{1}$$

this holds because, in general for any family $\langle C_i : i \in I\rangle$ of sets,

$$\coprod_{i \in I} C_i = \coprod_{i \in I} \{C_i : i \in I,\ C_i \text{ is inhabited}\}.$$

Continuing with the definition of $[\![F]\!]$, we put, for $(X,A) \in S$, $u \in |F|(X,A)$,

$$\sigma_{[\![F]\!]}\ ((X,A),u) = X, \qquad \tau_{[\![F]\!]}\ ((X,A),u) = A.$$

It follows by (1) that

$$|[\![F]\!]|\ (X,A) = \{((X,A),u) : u \in \| \, |F|(X,A)\| \}$$

for any $X \in \mathrm{Ob}(\boldsymbol{X})$, $A \in \mathrm{Ob}(\boldsymbol{A})$.

We define the natural transformation ι_F via a renaming transformation $\bar{\iota}_F : F \xrightarrow{\cong} [\![F]\!]$; the mapping

$$\bar{\iota}_F[X,A] : |F|(X,A) \xrightarrow{\cong} |[\![F]\!]|(X,Y)$$

is defined by

$$\bar{\iota}_F\ [X,A](v) = ((X,A), \iota_{|F|(X,A)}(v)),$$

with $\iota_{|F|(X,A)} : |F|(X,A) \to \| \, |F|(X,A)\|$ being given by the SCSA. The effect of $[\![F]\!]$ on morphisms is thereby determined: we put

$$[\![F]\!]_{\bar{s},\bar{t}}(f) = F_{s,t}(f)$$

whenever $f: X \to Y$, $s \in |F|\ X$, $t \in |F|\ Y$, $\bar{s} = \bar{\iota}_F\ [X,A](s)$, $\bar{t} = \bar{\iota}_F\ [Y,A](t)$. It is clear that the sanafunctor $[\![F]\!]$ and the natural isomorphism $\iota_F : F \xrightarrow{\cong} [\![F]\!]$ are thus well defined; and as we said above, it follows that $[\![F]\!] \in \mathrm{A}(\boldsymbol{X},\boldsymbol{A})$.

4. (SCSA) *For any small categories* $\boldsymbol{X},\boldsymbol{A}$, *there is a small full subcategory* $\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ *of* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *such that the inclusion* $\mathrm{A}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of categories.*

Proof. Combine 3 with 1.10.

The main result of this section is that, under the SCSA, AnaCat is Cartesian closed. In a natural way, in any bicategory, we say that A^X is an *exponential* of (X, A), with *evaluation* $e: X \times A^X \to A$, if for any object Y, the functor

$$(X \times (\ )) \circ e : \mathrm{Hom}(Y, A^X) \to \mathrm{Hom}(X \times Y, A)$$

$$Y \xrightarrow{f} A^X \mapsto (X \times Y \xrightarrow{X \times f} X \times A^X \xrightarrow{e} A)$$

is an equivalence of categories; a bicategory is *Cartesian closed* if it is Cartesian and has exponentials of all pairs of objects.

5. (SCSA) AnaCat *is Cartesian closed.*

Proof. By 1.18, Anacat is Cartesian, with the product structure computed as in Cat. 1.14 and 4 imply that exponentials exist; in fact, $\mathrm{A}(\boldsymbol{X},\boldsymbol{A})$ will act as an exponential $\boldsymbol{A}^{\boldsymbol{X}}$ in Anacat, with evaluation morphism $e: \boldsymbol{X} \times \mathrm{A}(\boldsymbol{X},\boldsymbol{A}) \to \boldsymbol{A}$ the restriction to $\boldsymbol{X} \times \boldsymbol{A}(\boldsymbol{X},\boldsymbol{A})$ of the map defined in (8), (8′) and (9) in section 1.

Since Sanacat is equivalent (as a bicategory) to AnaCat (1.12′), we have

6. (SCSA) SanaCat *is Cartesian closed.*

Terminal object, *product* and *exponentiation* in an anabicategory are defined by modifying the definitions for bicategories. In a anabicategory, we say that $A \xleftarrow{\pi} C \xrightarrow{\pi'} B$ is a *product diagram* if, for any object D, the anafunctor

$$(\pi \circ (-), \pi' \circ (=)) : \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, A) \times \mathrm{Hom}(D, B)$$

is an anaequivalence of categories. To emphasize the possibly obvious, here $\pi \circ (-) : \mathrm{Hom}(D, C) \to \mathrm{Hom}(D, A)$ is the section of the composition anafunctor

$$\circ_{D, C, A} : \mathrm{Hom}(D, C) \times \mathrm{Hom}(C, A) \to \mathrm{Hom}(D, A)$$

at $\pi \in \mathrm{Hom}(C, A)$; or what is the same, the composite of $\circ_{D, C, A}$ with

$$\mathrm{Hom}(D, C) \times \ulcorner \pi \urcorner : \mathrm{Hom}(D, C) \times \mathbf{1} \to \mathrm{Hom}(D, C) \times \mathrm{Hom}(C, A).$$

A^X, with $e: X \times A^X \to A$ (*evaluation*), is an *exponential of* (X, A) if for any Y, the anafunctor

$$(X \times (-)) \circ e : \mathrm{Hom}(Y, A^X) \to \mathrm{Hom}(X \times Y, A)$$

is an anaequivalence of categories.

T is a *terminal object* if, for any A, $\mathrm{Hom}(A, T) \to \mathbf{1}$, with $\mathbf{1}$ the one-object, one-arrow category, is an anaequivalenc of categories.

An anabicategory $\mathscr{A}$ is *Cartesian* if it has a terminal object, and binary products of arbitrary pairs of objects, and it is *Cartesian closed* if, in addition, it has exponentials. The definitions of the concepts involved are straightforward versions of the corresponding definitions for bicategories.

7. (SCSA) SanaCat$^{\#}$ *is Cartesian closed.*

The proof, whose details I leave to the reader, is based on the fact that the identity mapping is an equivalence of anabicategories between SanaCat$^{\#}$ and SanaCat, the latter understood as an anabicategory, and on the fact that the Cartesian closed character of an anabicategory is invariant under equivalences. Although the notion of "equivalence of anabicategories" has not been explicitly stated, the reader will not find it difficult to complete the proof.

Let me mention that the main result 5 holds with the SCSA replaced by Blass' SVC. The proof involves a weaker version of the SCSA, which is a consequence of the SVC, and suffices for the conclusion of 5. This weaker version of SCSA, will be discussed in the context of indexed category theory over a topos in [15]. The interest of the said variant of SCSA is heightened by the fact that it holds in all Grothendieck toposes, whereas direct topos-theoretic translates of SVC do not.

To end the paper, we will consider consequences of the Axiom of Regularity. Actually, only a consequence having the form of a weak version of the Axiom of Choice of the said axiom is used; this consequence I call the

Axiom of Hierarchy (AH). There is a family $\langle V_\alpha : \alpha \in \mathbf{Ord} \rangle$ of sets indexed by the ordinals whose union $\bigcup_{\alpha \in \mathbf{Ord}} V_\alpha$ is the class of all sets.

As is well known, the AH is made true by the Axiom of Regularity through the von-Neumann hierarchy $\langle V_\alpha : \alpha \in \mathbf{Ord} \rangle$ of pure (regular) sets. In what follows, we will assume the AH, and also the validity of classical logic.

The AH gives a metafunction assigning to any class A a subset $\bar{A} \subset A$ such that if A is non-empty, then so is $\bar{A}$; if A is non-empty, $\bar{A} = A \cap V_\alpha$ for α the least α for which $A \cap V_\alpha \neq \emptyset$, $\bar{A} = \emptyset$ otherwise. Combined with the GAC, we have a meta-choice-function, assigning to every non-empty class an element of it. Therefore, using the proof of 1.11, we can conclude the following strengthening of 1.11:

8. (GAC, AH) *The inclusion* $\mathrm{FUN}(\mathbf{X}, \mathbf{A}) \to \mathrm{ANA}(\mathbf{X}, \mathbf{A})$ *is an equivalence of meta-categories.*

(This was observed by the referee, correcting a careless statement I originally made.)

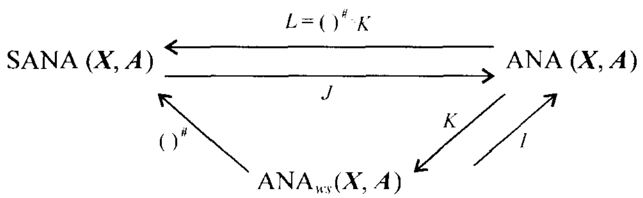
9. (AH,CL) *The inclusion* $\mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories. In particular, if* $\boldsymbol{X}$ *is a small category, the inclusion* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories.*

Indeed, given any $F \in \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$, define $\bar{F} \in \mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ by $|\bar{F}|X = (|F|X)^{-}$ (with the latter $(\ )^{-}$ understood as in $A \mapsto \bar{A}$ above), and otherwise restricting the data of F appropriately; 1.1 (iii) for $\bar{F}$ is ensured by the construction, and all other laws of "anafunctor" are automatically true for $\bar{F}$. We have the natural isomorphism $h: \bar{F} \xrightarrow{\cong} F$ whose components $h_{s,s}$ for $s \in |\bar{F}|(X,A)$ ($\subset |F|(X,A)$) are identities.

We can now generalize, under the AH, 1.10 to obtain

10. (AH, CL) *The inclusion* $\mathrm{SANA}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *is an equivalence of metacategories, provided* $\boldsymbol{A}$ *is locally small.*

To see this, note that in the proof of 1.10, the construction $F^{\#}$ is legitimate provided $\boldsymbol{A}$ is locally small, and $F: \boldsymbol{X} \xrightarrow{a} \boldsymbol{A}$ is weakly small; the class $S_{X,A}$ introduced there is a set in this case, and we may consider the set of equivalence classes of the equivalence relation $\sim$ on $S_{X,A}$ as needed there. (Note, however, that $F^{\#}$ constructed is not necessarily weakly small.) We obtain the functor $(\ )^{\#}: \mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A}) \to \mathrm{SANA}(\boldsymbol{X},\boldsymbol{A})$ such that for every $F \in \mathrm{ANA}_{ws}(\boldsymbol{X},\boldsymbol{A})$, $F \cong F^{\#}$ with a canonical isomorphism; that is, in



where I and J are inclusions, and K is the quasi-inverse of I given by 9, we have $J \circ (\ )^{\#} \cong I$. But then

$$J \circ L \cong J \circ (\ )^{\#} \circ K \cong I \circ K \cong \mathrm{Id}, \tag{2}$$

and

$$J \circ L \circ J = J \circ (\ )^{\#} \circ K \circ J \cong I \circ K \circ J \cong J = J \circ \mathrm{Id},$$

which, since J is full and faithful, implies that

$$L \circ J \cong \mathrm{Id}. \tag{3}$$

(2) and (3) show what we want.

As an immediate consequence of 9 and 2.6, we have

11. (AH, CL) *Suppose that the category* $\boldsymbol{A}$ *has small limits, and* $\boldsymbol{X}$ *is a small category. Then* $\mathrm{ANA}(\boldsymbol{X},\boldsymbol{A})$ *and* $\mathrm{Ana}(\boldsymbol{X},\boldsymbol{A})$ *have specified small limits.*

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Note added in proof

After the completion of this paper, G.M. Kelly drew my attention to his paper "Complete functors in homology I. Chain maps and endomorphisms" (Proc. Cambridge Phil. Soc. 60 (1964) 721-735). In Section 2, "Generalities on functors", on page 723, he gives a concept, without naming it, which is identical to that of "anafunctor". He gives the definition in both forms as in 1.1 and 1.1* of the present paper. He presents the concept as the general form the available data for a functor often take; in such cases, converting the data into a functor requires the class-form of the Axiom of Choice. He does not attempt to develop a theory of such data; however, he does say: "one who will not admit such choices [requiring the Axiom of Choice] may work with the pair of honest functors S, T [in the span-style definition of "anafunctor"] in place of the dishonest functor . . .". Thus, the present paper is a working-out of a thirty-year old idea of Max Kelly's.

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